A New Sufficient Condition for Complete Stability of Cellular Neural Networks with Delay

Author(s): Norikazu Takahashi

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Norikazu Takahashi, *Member, IEEE*

Abstract— **This paper gives a new sufficient condition for Cellular Neural Networks with Delay (DCNN's) to be completely stable. A fixed-point theorem and a convergence theorem of the Gauss-Seidel method play important roles in the proof, while most conventional stability criteria were obtained by constructing Lyapunov functionals.**

Index Terms— **Cellular neural networks, Delay, Complete stability, Gauss-Seidel method**

I. INTRODUCTION

Cellular neural networks (CNN's) introduced by Chua and Yang [1] have many applications in the area of image processing [2], [3]. Since CNN's are usually required to be completely stable for their applications, studies on complete stability of CNN's have been vigorously done and many criteria have been obtained so far [1], [4]– [9].

As a generalization of the standard CNN model, Cellular Neural Networks with Delay (DCNN's) were introduced by Roska and Chua [10]. In DCNN's, the state transition of each cell depends not only on the present outputs of its neighbor cells but also on the delayed outputs of them. Because of the delay effect, DCNN's are thought to be useful for motion-related image processing. However, the dynamical behavior of DCNN's is so complicated that studies on complete stability of DCNN's are much more difficult than that of standard CNN's.

So far, some complete stability criteria for DCNN's have been obtained [7], [11], [12]. In [11], it was proved that DCNN's with positive cell-linking templates are completely stable almost everywhere. In [12], it was shown that if sum of the feedback matrix and the delayed feedback matrix is symmetric and the length of delay is smaller than a certain value depending on the delayed feedback matrix then the DCNN is completely stable. The stability condition given in [7] is similar to the one in [12]. Besides these complete stability criteria, some sufficient conditions for a DCNN to have a globally asymptotically stable equilibrium point have also been obtained [13]– [15].

It should be noted that the existing stability criteria for standard CNN's can not necessarily be applied to DCNN's. It was in fact reported in [12] that a DCNN can become unstable even though both the feedback template and the delayed feedback template are symmetric.

In this paper, we give a new sufficient condition for a DCNN to be completely stable. The main result of the present paper is based on the stability condition for standard CNN's given in [9]. A convergence theorem of the Gauss-Seidel method [16], [17], which is an iterative technique for solving linear

The author is with the Department of Informatics, Kyushu University, Fukuoka, Japan.

algebraic equations, and a fixed-point theorem play important roles. We also show that a conjecture concerning the complete stability of DCNN's given in [14] does not hold true. A counterexample to the conjecture will be given.

II. DCNN MODEL

The state equation of a DCNN is described by the following differential equation

$$\frac{d\boldsymbol{x}(t)}{dt} = -\boldsymbol{x}(t) + \boldsymbol{A}\boldsymbol{y}(t) + \boldsymbol{A}^{\tau}\boldsymbol{y}(t-\tau) + \boldsymbol{b} \qquad (1)$$

where $\boldsymbol{x}(t) = [x_1(t), \ldots, x_n(t)]^T$ is the state, $\boldsymbol{y}(t) = [y_1(t), \ldots, y_n(t)]^T$ is the output, $\boldsymbol{b} = [b_1, \ldots, b_n]^T$ is the input, $\boldsymbol{A} = [a_{ij}]$ is the feedback matrix, $\boldsymbol{A}^{\tau} = [a_{ij}^{\tau}]$ is the delayed feedback matrix, τ is a positive number representing the length of delay, and n is the number of cells. The value of each element y_i of $\boldsymbol{y}$ is determined from x_i as follows:

$$y_i(t) = f(x_i(t)), \quad i = 1, 2, \ldots, n$$

where $f(\cdot)$ is the piecewise linear function defined by

$$f(v) = \frac{1}{2}(|v+1| - |v-1|).$$

The initial condition for a DCNN is given by

$$\boldsymbol{x}(t) = \boldsymbol{\phi}^0(t), \quad t \in [-\tau, 0] \qquad (2)$$

where $\boldsymbol{\phi}^0(t)$ is assumed to be a continuous function on $[-\tau, 0]$. One should note that the DCNN model includes the standard CNN as the special case where $\boldsymbol{A}^{\tau} = \boldsymbol{0}$.

Definition 1: A DCNN is said to be completely stable if and only if the trajectory $\boldsymbol{\phi}(t, \boldsymbol{\phi}^0)$ of the state equation (1) satisfies

$$\lim_{t \to \infty} \boldsymbol{\phi}(t, \boldsymbol{\phi}^0) = \text{const.} \qquad (3)$$

holds for any continuous function $\boldsymbol{\phi}^0(t)$.

If a DCNN is not completely stable, it has at least one trajectory $\boldsymbol{\phi}(t, \boldsymbol{\phi}^0)$ which does not satisfy (3). In the following, such a trajectory is said to be unstable and denoted with the superscript u, as $\boldsymbol{\phi}^u(t, \boldsymbol{\phi}^0)$.

We now give a few definitions which will be needed in the following sections.

Definition 2: An $n \times n$ matrix $\boldsymbol{P}$ with nonpositive off-diagonal elements is called M-matrix if all its principal minors are positive.

Definition 3: Let $\boldsymbol{P}$ be a matrix with positive diagonal elements. The comparison matrix $\boldsymbol{S}$ of $\boldsymbol{P}$ is defined as $s_{ii} = p_{ii}$ and $s_{ij} = -|p_{ij}|$ for $i \neq j$.

III. Main Result and Related Works

The following theorem, which gives a new sufficient condition for DCNN's to be completely stable, is the main result of this paper. The proof will be given in Section IV.

Theorem 1: Let $W = [w_{ij}]$ be the $n \times n$ matrix determined from the matrices A and A^τ as follows:

$$w_{ij} = \begin{cases} a_{ii} - 1 - |a_{ii}^\tau| & \text{if } i = j \\ -|a_{ij}| - |a_{ij}^\tau| & \text{otherwise} \end{cases} . \tag{4}$$

If W is an M-matrix, then the DCNN is completely stable.

This is based on the following theorem.

Theorem 2 ([9]): A standard CNN is completely stable if the comparison matrix of $A - I$ is an M-matrix where I denotes the identity matrix.

Since a standard CNN can be regarded as a DCNN satisfying $A^\tau = 0$, Theorem 2 corresponds to the special case of Theorem 1 where $A^\tau = 0$. The statement of Theorem 2 was firstly given in [8] as a conjecture, and then proved to be true in [9]. A convergence theorem of the Gauss-Seidel method plays a key role in the proof of Theorem 2. As we will see later, this technique is also efficient for the proof of Theorem 1.

Theorem 1 is also a generalization of the following theorem because a row sum dominant matrix is an M-matrix.

Theorem 3 ([7]): A DCNN is completely stable if the matrix W is row sum dominant.

The sufficient condition in Theorem 1 is distinguished from the existing complete stability criteria [7], [11], [12] on the following points: i) the condition does not impose any restriction on signs of the off-diagonal elements of A and A^τ, and ii) the condition is independent of the length of delay.

IV. Proof of Theorem 1

In this section, we give a proof of Theorem 1. First, the complete stability of one-cell CNN's is investigated, and then DCNN's with arbitrary number of cells will be dealt with.

A. One-Cell DCNN's

Let us consider the dynamics of DCNN's consisting of only one cell:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -x(t) + ay(t) + a^\tau y(t - \tau) + b \tag{5}$$

with the initial condition

$$x(t) = \phi^0(t), \quad \forall t \in [-\tau, 0].$$

By putting

$$g(t) = a^\tau y(t - \tau) + b \tag{6}$$

we can rewrite (5) as follows:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -x(t) + af(x(t)) + g(t) \tag{7}$$

Now we introduce the DP plot [3] which will be needed in some of the lemmas given below. The DP plot $\Gamma(t)$ is the curve obtained by plotting the right-hand side of (7) as a function of $x(t)$. Fig.1 shows a DP plot $\Gamma(t)$ for certain values of a and $g(t)$. One should note that $\Gamma(t)$ moves up and down as time increases since the value of $g(t)$ varies with time.
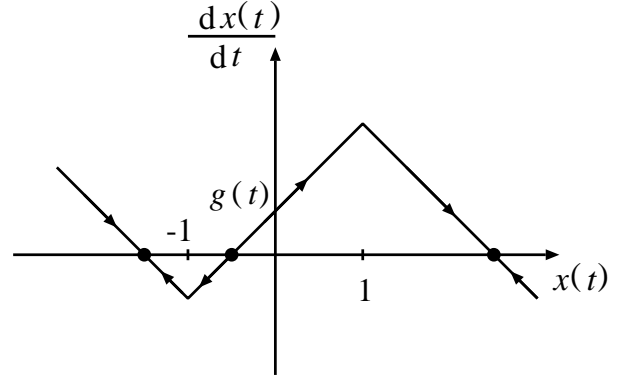

Fig. 1. DP plot $\Gamma(t)$

Lemma 1: If $a > 1$ then any unstable trajectory $\phi^u(t, \phi^0)$ of (5) satisfies

$$f(\phi^u(t, \phi^0)) \in I^{(k)}, \quad \forall t \geq (k-1)\tau \tag{8}$$

for all $k \ (= 0, 1, 2, \ldots)$, where $I^{(k)}$ is the closed interval obtained by the following algorithm with $u^{(0)} = 1$ and $l^{(0)} = -1$.

Algorithm 2

   Step 1: Set $I^{(0)} = [l^{(0)}, u^{(0)}]$.
   Step 2: Set $k = 1$.
   Step 3: Compute

$$u^{(k)} = f\left(-\frac{\min_{y \in I^{(k-1)}}\{a^\tau y\} + b}{a - 1}\right) \tag{9}$$

$$l^{(k)} = f\left(-\frac{\max_{y \in I^{(k-1)}}\{a^\tau y\} + b}{a - 1}\right) \tag{10}$$

   and set $I^{(k)} = [l^{(k)}, u^{(k)}]$.
   Step 4: Add 1 to k, and go to Step 3.

Proof: It is obvious that (8) holds for $k = 0$ since $I^{(0)} = [-1, 1]$. In the following, we will show that if (8) holds for $k = k'(\geq 0)$, that is,

$$f(\phi^u(t, \phi^0)) \in I^{(k')}, \quad \forall t \geq (k' - 1)\tau \tag{11}$$

then it also holds for $k = k' + 1$.

It follows from (11) that $g(t)$ is bounded as $g_{\min} \leq g(t) \leq g_{\max}$ for any unstable trajectory and for all $t \geq k'\tau$, where

$$g_{\min} = \min_{y \in I^{(k')}}\{a^\tau y\} + b \tag{12}$$

$$g_{\max} = \max_{y \in I^{(k')}}\{a^\tau y\} + b. \tag{13}$$

Let $\Gamma_{\max}$ and $\Gamma_{\min}$ denote the DP plots corresponding to the cases where $g(t) = g_{\max}$ and $g(t) = g_{\min}$, respectively. Then the DP plot $\Gamma(t)$ of any unstable trajectory is bounded by $\Gamma_{\max}$ and $\Gamma_{\min}$ for $t \geq k'\tau$ as shown in Fig.2. In Fig.2, p (resp., q) denotes the value of x at the point where the middle segment of $\Gamma_{\min}$ (resp., $\Gamma_{\max}$) and the x-axis intersect. The values of
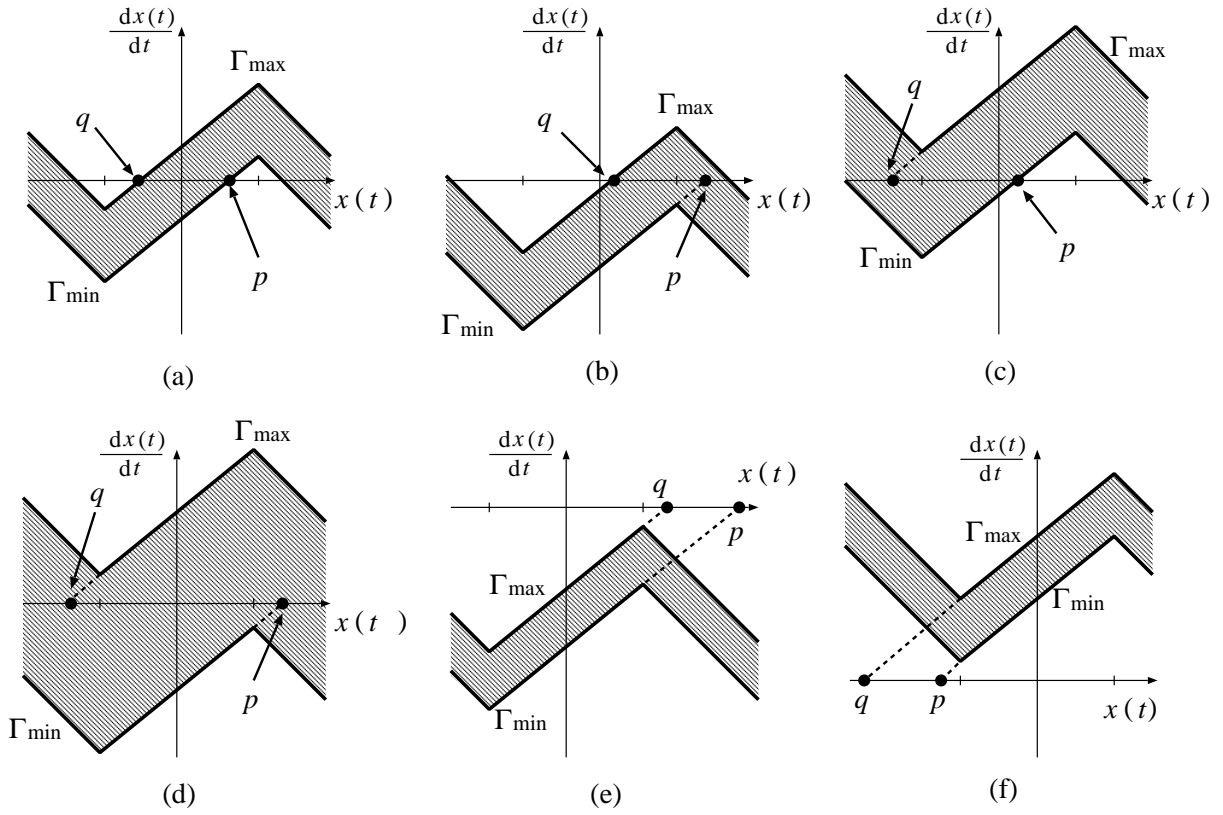
Fig. 2. The region bounded by $\Gamma_{\max}$ and $\Gamma_{\min}$. These figures correspond to the six possible cases depending the value of p and q: (a) $-1 < q \leq p < 1$, (b) $-1 < q < 1 \leq p$, (c) $q \leq -1 < p < 1$, (d) $q \leq -1 < 1 \leq p$, (e) $1 \leq q \leq p$, (f) $q \leq p \leq -1$

p and q are expressed by the following equations:

$$p = -\frac{g_{\min}}{a-1} = -\frac{1}{a-1}\left(\min_{y \in I^{(k')}}\{a^\tau y\} + b\right) \quad (14)$$

$$q = -\frac{g_{\max}}{a-1} = -\frac{1}{a-1}\left(\max_{y \in I^{(k')}}\{a^\tau y\} + b\right). \quad (15)$$

Fig.2 shows six possible cases depending on the values of p and q. In Fig.2(a), $dx(t)/dt$ is positive for any value of $g(t)$ as long as $x(t)$ satisfies $p < x(t) \leq 1$. Therefore, if $p < \phi^u(t, \phi^0) \leq 1$ holds for some $t \geq k'\tau$ then $\lim_{t \to \infty} f(\phi^u(t, \phi^0)) = 1$ holds. This implies that $\phi^u(t, \phi^0)$ converges to an equilibrium point. Also, if $-1 \leq \phi^u(t, \phi^0) < q$ holds for some $t \geq k'\tau$ then $\phi^u(t, \phi^0)$ converges to an equilibrium point. Therefore, any unstable trajectory $\phi^u(t, \phi^0)$ must satisfy

$$q \leq \phi^u(t, \phi^0) \leq p, \quad \forall t \geq k'\tau$$

In Figs.2(b) and (e), any unstable trajectory $\phi^u(t, \phi^0)$ must satisfy

$$q \leq \phi^u(t, \phi^0), \quad \forall t \geq k'\tau$$

because if $\phi^u(t, \phi^0) < q$ for some t then we have $\lim_{t \to \infty} f(\phi^u(t, \phi^0)) = -1$ which implies $\phi^u(t, \phi^0)$ converges to an equilibrium point. Similarly, in Fig.2(c) and (f), any unstable trajectory $\phi^u(t, \phi^0)$ must satisfy

$$\phi^u(t, \phi^0) \leq p, \quad \forall t \geq k'\tau$$

because if $\phi^u(t, \phi^0) > p$ for some t then it converges to an equilibrium point. Finally, in the case of Fig.2(d), there is no restriction on the value of $\phi^u(t, \phi^0)$ for $t \geq k'\tau$.

Taking all the above cases into account, we can conclude that any unstable trajectory $\phi^u(t, \phi^0)$ satisfies

$$f(q) \leq f(\phi^u(t, \phi^0)) \leq f(p), \quad \forall t \geq k'\tau. \quad (16)$$

Since $f(p) = u^{(k'+1)}$ and $f(q) = l^{(k'+1)}$, (16) can be rewritten as follows:

$$f(\phi^u(t, \phi^0)) \in I^{(k'+1)}, \quad \forall t \geq k'\tau.$$

Therefore (8) holds for $k = k' + 1$. ∎

Lemma 2: If a and a^τ satisfy

$$a - 1 - |a^\tau| > 0 \quad (17)$$

then the sequences $\{u^{(k)}\}$ and $\{l^{(k)}\}$ obtained by Algorithm 2 converge to the same value, that is,

$$\lim_{k \to \infty} u^{(k)} = \lim_{k \to \infty} l^{(k)} = c. \quad (18)$$

Proof: See Appendix. ∎

The following lemma follows from Lemmas 1 and 2.

Lemma 3: The one-cell DCNN (5) is completely stable if a and a^τ satisfy (17).

Proof: From Lemma 1, any unstable trajectory $\phi^u(t, \phi^0)$ of (5) satisfies (8) for all $k(= 0, 1, 2, \cdots)$. However, if a and a^τ satisfy (17), it follows from Lemma 2 that

$$\lim_{t \to \infty} f(\phi^u(t, \phi^0)) = \text{const.}$$

which implies $\phi^u(t, \phi^0)$ converges to an equilibrium point. This is a contradiction. ∎

Now we consider one-cell DCNN's with time-varying input:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -x(t) + ay(t) + a^\tau y(t-\tau) + b(t) \qquad (19)$$

where $b(t)$ is assumed to be bounded as

$$b_{\min} \le b(t) \le b_{\max}, \quad \forall t \ge 0.$$

Although this is not the model of interest in this paper, the dynamic properties of (19) are useful in later discussion.

Lemma 4: If $a > 1$ then any unstable trajectory $\phi^u(t, \phi^0)$ of (19) satisfies

$$f(\phi^u(t, \phi^0)) \in I^{(k)}, \quad \forall t \ge (k-1)\tau$$

for all k $(= 0, 1, 2, \ldots)$, where $I^{(k)}$ is the closed intervals given by the following algorithm with $u^{(0)} = 1$ and $l^{(0)} = -1$.

Algorithm 3

Step 1: Set $I^{(0)} = [l^{(0)}, u^{(0)}]$.

Step 2: Set $k = 1$.

Step 3: Compute

$$u^{(k)} = f\left(-\frac{\min_{y \in I^{(k-1)}}\{a^\tau y\} + b_{\min}}{a-1} \right)$$

$$l^{(k)} = f\left(-\frac{\max_{y \in I^{(k-1)}}\{a^\tau y\} + b_{\max}}{a-1} \right)$$

and set $I^{(k)} = [l^{(k)}, u^{(k)}]$.

Step 4: Add 1 to k, and go to Step 3.

We omit the proof of Lemma 4 because it is very similar to that of Lemma 1.

Lemma 5: If a and a^τ satisfy (17) then the sequence $\{I^{(k)}\}$ obtained by Algorithm 3 converges to the closed interval $I^* = [l^*, u^*] \subset [-1, 1]$ which is uniquely determined from a, a^τ, $b_{\max}$ and $b_{\min}$. Moreover, the interval I^* satisfies

$$u^* - l^* \le \frac{b_{\max} - b_{\min}}{a - 1 - |a^\tau|} \qquad (20)$$

Proof: See Appendix. ∎

B. DCNN's with arbitrary number of cells

Lemma 6: Let $\boldsymbol{A}_i$ and $\boldsymbol{A}_i^\tau$ denote the $(n-1) \times (n-1)$ matrices obtained by eliminating the i-th row and the i-th column from $\boldsymbol{A}$ and $\boldsymbol{A}^\tau$, respectively. If every $(n-1)$-cell DCNN defined by

$$\frac{\mathrm{d}\boldsymbol{x}(t)}{\mathrm{d}t} = -\boldsymbol{x}(t) + \boldsymbol{A}_i \boldsymbol{y}(t) + \boldsymbol{A}_i^\tau \boldsymbol{y}(t-\tau) + \boldsymbol{b} \qquad (21)$$

is completely stable, and if

$$a_{ii} - 1 - |a_{ii}^\tau| > 0, \quad i = 1, 2, \ldots, n \qquad (22)$$

holds, then any unstable trajectory $\phi^u(t, \phi^0)$ of the n-cell DCNN (1) satisfies

$$\lim_{t\to\infty} f(\phi^u(t, \phi^0)) \in D^{(m)} \qquad (23)$$

for $m = 0, 1, 2, \ldots$ where $D^{(m)}$ is the closed region determined by Algorithm 4 below.

Algorithm 4

Step 1: Set

$$D_i^{(0)} = [-1, 1], \quad i = 1, 2, \ldots, n$$

and

$$D^{(0)} = D_1^{(0)} \times D_2^{(0)} \times \cdots \times D_n^{(0)}.$$

Step 2: Set $m = 1$.

Step 3: Determine $D_i^{(m)}$ $(i = 1, 2, \ldots, n)$ by the following substeps.

Substep 3.1: Set $i = 1$.

Substep 3.2: Set

$$I^{(0)} = D_i^{(m-1)} \qquad (24)$$

and

$$
\begin{aligned}
b_{\min} &= \sum_{j=1}^{i-1} \min_{y_j, y_j^\tau \in D_j^{(m)}} \{a_{ij} y_j + a_{ij}^\tau y_j^\tau\} \\
&\quad + \sum_{j=i+1}^{n} \min_{y_j, y_j^\tau \in D_j^{(m-1)}} \{a_{ij} y_j + a_{ij}^\tau y_j^\tau\} + b_i \quad (25)
\end{aligned}
$$

$$
\begin{aligned}
b_{\max} &= \sum_{j=1}^{i-1} \max_{y_j, y_j^\tau \in D_j^{(m)}} \{a_{ij} y_j + a_{ij}^\tau y_j^\tau\} \\
&\quad + \sum_{j=i+1}^{n} \max_{y_j, y_j^\tau \in D_j^{(m-1)}} \{a_{ij} y_j + a_{ij}^\tau y_j^\tau\} + b_i. \quad (26)
\end{aligned}
$$

Substep 3.3: Execute Algorithm 3 with (24)–(26) until $\{I^{(k)}\}$ converges to a certain closed interval I^*.

Substep 3.4: Set

$$D_i^{(m)} = I^*.$$

Substep 3.5: If $i = n$, go to *Step 4*. Otherwise, add 1 to i and go to *Substep 3.2*.

Step 4: Set

$$D^{(m)} = D_1^{(m)} \times D_2^{(m)} \times \cdots \times D_n^{(m)}.$$

Step 5: Add 1 to m and go to *Step 3*.

Proof: It is obvious that (23) holds for $m = 0$. We will show in the following that if (23) holds for $m = m'$, then it also holds for $m = m' + 1$. First we rewrite the i-th equation of (1) as

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -x_i(t) + a_{ii} y_i(t) + a_{ii}^\tau y_i(t-\tau) + b_i(t) \qquad (27)$$

where

$$b_i(t) = \sum_{j=1, j\neq i}^{n} (a_{ij} y_j(t) + a_{ij}^\tau y_j(t-\tau)) + b_i$$

Since

$$\lim_{t\to\infty} f(\phi^u(t, \phi^0)) \in D^{(m')} \qquad (28)$$

holds from the assumption, $b_i(t)$ eventually satisfies

$$b_{i,\min} \le b_i(t) \le b_{i,\max}$$

where

$$b_{i,\max} = \sum_{j=1, j\neq i}^{n} \left[\max_{y_j, y_j^\tau \in D_j^{(m')}} \{a_{ij}y_j + a_{ij}^\tau y_j^\tau\} \right] + b_i$$

$$b_{i,\min} = \sum_{j=1, j\neq i}^{n} \left[\min_{y_j, y_j^\tau \in D_j^{(m')}} \{a_{ij}y_j + a_{ij}^\tau y_j^\tau\} \right] + b_i.$$

One should note that (27) is regarded as a one-cell DCNN with time-varying input. One should also note that each element $\phi_i^u(t, \phi^0)$ of an unstable trajectory $\phi^u(t, \phi^0)$ must be unstable, because if there exists an i such that $\lim_{t\to\infty} \phi_i^u(t, \phi^0) = $ const. then the state equation at $t = \infty$ can be written in the form of (21) which implies $\phi^u(t, \phi^0)$ converges to an equilibrium point.

By applying Lemma 4 to $\phi_1^u(t, \phi^0)$, we have

$$\lim_{t\to\infty} \phi_1^u(t, \phi^0) \in I^* = D_1^{(m'+1)}$$

where I^* is the limit of the sequence $\{I^{(k)}\}$ obtained by Algorithm 3 with $I^{(0)} = D_1^{(m')}$, $b_{\max} = b_{1,\max}$, and $b_{\min} = b_{1,\min}$. Hence (28) can be updated to

$$\lim_{t\to\infty} \phi^u(t, \phi^0) \in D_1^{(m'+1)} \times D_2^{(m')} \times \cdots \times D_n^{(m')}.$$

Similarly, applying Lemma 4 to ϕ_2^u, ϕ_3^u, ..., ϕ_n^u, we have

$$\lim_{t\to\infty} \phi^u(t, \phi^0) \in D_1^{(m'+1)} \times \cdots \times D_n^{(m'+1)} = D^{(m'+1)}.$$

Therefore, (23) holds for $m = m' + 1$. ∎

Lemma 7: If $\boldsymbol{W}$ is an M-matrix, the sequence $\{D^{(m)}\}$ obtained by Algorithm 4 converges to a singleton.

Proof: See Appendix. ∎

Now we are ready to prove Theorem 1.

Proof of Theorem 1: It is obvious from Lemma 3 that the theorem holds for one-cell DCNN's. Assume that the theorem holds for any $(n-1)$-cell DCNN ($n \geq 2$). We will show that the n-cell DCNN such that $\boldsymbol{W}$ is an M-matrix is completely stable under this assumption.

Since any principal submatrix of an M-matrix is also an M-matrix, any $(n-1)$-cell DCNN defined by (21) is completely stable. Moreover, since $\boldsymbol{W}$ is an M-matrix, (22) holds. Thus Lemma 6 can be applied to the n-cell DCNN, and we can say that any unstable trajectory $\phi^u(t, \phi^0)$ satisfies (23). However, it follows from Lemma 7 that $\lim_{m\to\infty} D^{(m)}$ is a singleton which implies $\lim_{t\to\infty} \phi^u(t, \phi^0) = $ const. This is a contradiction. Therefore, there is no unstable trajectory. In other words, the n-cell DCNN is completely stable. ∎

V. A CONJECTURE ON THE COMPLETE STABILITY OF DCNN'S

Concerning the complete stability of DCNN's, the following conjecture was recently made by Arik and Tavsanoglu.

Conjecture 1 ([14]): A DCNN is completely stable if the comparison matrix of $\boldsymbol{A} + \boldsymbol{A}^\tau - \boldsymbol{I}$ is an M-matrix.

It should be noted that Theorem 2 corresponds to the special case of Conjecture 1 where $\boldsymbol{A}^\tau = \boldsymbol{0}$. In addition, Conjecture 1 is closely related to Theorem 1. In fact, the class of completely stable DCNN's given in Theorem 1 is strictly included in that of Conjecture 1.

Conjecture 1 is based on the fact that if the comparison matrix of $\boldsymbol{A} + \boldsymbol{A}^\tau - \boldsymbol{I}$ is an M-matrix then only the total saturation region contains stable equilibrium points. However, no one has proved so far that it is true or false.

We now show that Conjecture 1 does not hold true by giving a counterexample. Let us consider the following one-cell DCNN:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -x(t) + y(t) + y(t - \tau). \qquad (29)$$

This DCNN satisfies the sufficient condition for complete stability given in Conjecture 1 because $\boldsymbol{A} + \boldsymbol{A}^\tau - \boldsymbol{I} = a_{11} + a_{11}^\tau - 1 = 1 > 0$ holds. Now let $\tau = 3\pi/2$ and $\phi^0(t) = r\sin(t)$ $(-\tau \leq t \leq 0)$ where $0 < r < 1$. Then, for $x(t) = r\sin(t)$, we have

$$\begin{aligned} &-x(t) + y(t) + y(t - \tau) \\ &= -r\sin(t) + r\sin(t) + r\sin(t - 3\pi/2) \\ &= r\cos(t). \end{aligned}$$

Thus $x(t) = r\sin(t)$ is a solution of the differential equation (29). Since $x(t) = r\sin(t)$ never converges to an equilibrium point, the above DCNN is not completely stable.

VI. CONCLUDING REMARKS

We have derived a new sufficient condition for DCNN's to be completely stable by utilizing a fixed-point theorem and a convergence theorem of the Gauss-Seidel method to the stability analysis of DCNN's. Also, we have shown that the conjecture made in [14] does not hold by giving a counterexample. It should be noted that the complete stability criterion given in this paper is independent of the length of delay. In addition, the criterion is valid even though the length of delay differs from cell to cell.

However, as shown in [9], the new condition is rather severe, particularly for DCNN's with space-invariant templates. Finding new classes of completely stable DCNN's which provide us useful applications is a future problem.

APPENDIX

Proof of Lemma 2: Case 1) Suppose $a^\tau < 0$. Since (9) and (10) become

$$u^{(k)} = f\left(-\frac{a^\tau u^{(k-1)} + b}{a - 1}\right)$$

$$l^{(k)} = f\left(-\frac{a^\tau l^{(k-1)} + b}{a - 1}\right)$$

respectively, the sequences $\{u^{(k)}\}$ and $\{l^{(k)}\}$ are independently obtained. Let

$$h(v) = f\left(-\frac{a^\tau v + b}{a - 1}\right).$$

Then it is obvious that if v belongs to the closed interval $[-1, 1]$ then so does $h(v)$. Moreover, we have

$$
\begin{aligned}
|h(v) - h(v')| &= \left| f\left(-\frac{a^\tau v + b}{a-1}\right) - f\left(-\frac{a^\tau v' + b}{a-1}\right) \right| \\
&\leq \left| \left(-\frac{a^\tau v + b}{a-1}\right) - \left(-\frac{a^\tau v' + b}{a-1}\right) \right| \\
&= \left| \frac{-a^\tau}{a-1}(v - v') \right| \\
&= \frac{-a^\tau}{a-1}|v - v'|.
\end{aligned}
$$

Since $0 \leq -a^\tau/(a-1) < 1$ holds from (17), $h(v) : [-1, 1] \to [-1, 1]$ is a contraction mapping. From a fixed-point theorem, the equation $v = h(v)$ has a unique solution in $[-1, 1]$ and both $\{u^{(k)}\}$ and $\{l^{(k)}\}$ converge to the unique solution.

Case 2) Suppose $a^\tau > 0$. Then (9) and (10) become

$$
u^{(k)} = f\left(-\frac{a^\tau l^{(k-1)} + b}{a-1}\right) \tag{30}
$$

$$
l^{(k)} = f\left(-\frac{a^\tau u^{(k-1)} + b}{a-1}\right) \tag{31}
$$

respectively. Let $v = [u, l]^T$, and

$$
h(v) = \begin{bmatrix} h_1(v) \\ h_2(v) \end{bmatrix} = \begin{bmatrix} f(-\frac{a^\tau l + b}{a-1}) \\ f(-\frac{a^\tau u + b}{a-1}) \end{bmatrix}.
$$

It is obvious that if $v \in [-1, 1]^2 (= [-1, 1] \times [-1, 1])$ then $h(v) \in [-1, 1]^2$. Moreover, one can easily derive

$$
|h_1(v) - h_1(v')| \leq \frac{a^\tau}{a-1}|l - l'|
$$

and

$$
|h_2(v) - h_2(v')| \leq \frac{a^\tau}{a-1}|u - u'|.
$$

Thus we have

$$
||h(v) - h(v')||_\infty \leq \frac{a^\tau}{a-1}||v - v'||_\infty.
$$

Since $0 \leq a^\tau/(a-1) < 1$ holds from (17), $h(v) : [-1, 1]^2 \to [-1, 1]^2$ is a contraction mapping. From a fixed-point theorem, the equation $v = h(v)$ has a unique solution in $[-1, 1]^2$ and the sequence $\{v^{(k)} = [u^{(k)}, l^{(k)}]^T\}$ converges to the unique solution. By taking the symmetry of (30) and (31) into account, we derive (18). ∎

Proof of Lemma 5: We only give a proof of the second statement because the first one can be proved in a similar way as Lemma 2. Since $v^* = [u^*, l^*]^T$ is the fixed-point of Algorithm 3, u^* and l^* satisfy

$$
u^* = f\left(-\frac{\min_{y \in [l^*, u^*]}\{a^\tau y\} + b_{\min}}{a-1}\right)
$$

and

$$
l^* = f\left(-\frac{\max_{y \in [l^*, u^*]}\{a^\tau y\} + b_{\max}}{a-1}\right).
$$

Thus we have

$$
u^* - l^* \leq \frac{|a^\tau|(u^* - l^*) + (b_{\max} - b_{\min})}{a-1}
$$

which can be rewritten as (20). ∎

Proof of Lemma 7: Let $|D_i^{(m)}|$ denote the length of the interval $D_i^{(m)}$, and let $d^{(m)} = [d_1^{(m)}, \ldots, d_n^{(m)}]^T = [|D_1^{(m)}|, \ldots, |D_n^{(m)}|]^T$. Since it is obvious that $D^{(0)} \supseteq D^{(1)} \supseteq D^{(2)} \supseteq \cdots$ holds, it suffices for us to show that $\{d^{(m)}\}$ converges to the zero vector.

Let $\{z^{(m)}\}$ be the sequence of vectors which is obtained by the following algorithm.

Algorithm 5
   Step 1: Set $m = 1$ and $z_i^{(0)} = 2$ $(i = 1, 2, \ldots, n)$.
   Step 2: For $i = 1, 2, \ldots, n$, compute

$$
z_i^{(m)} = \frac{\sum_{j=1}^{i-1}(|a_{ij}| + |a_{ij}^\tau|)z_j^{(m)} + \sum_{j=i+1}^{n}(|a_{ij}| + |a_{ij}^\tau|)z_j^{(m-1)}}{a_{ii} - 1 - |a_{ii}^\tau|}
$$

and set

$$
z^{(m)} = [z_1^{(m)}, z_2^{(m)}, \ldots, z_n^{(m)}]^T.
$$

   Step 3: Add 1 to m and go to Step 2.

Algorithm 5 describes the Gauss-Seidel method for solving the equation $Wz = 0$ with the initial approximation $z^{(0)} = [2, 2, \ldots, 2]^T$. It is well known that the Gauss-Seidel method for solving a linear algebraic equation $Pz = b$ converges to the true solution if P is an M-matrix [16]. Therefore the sequence $\{z^{(m)}\}$ converges to the zero vector.

From the initial condition of Algorithms 4 and 5, we have

$$
d_i^{(0)} = z_i^{(0)}, \quad i = 1, 2, \ldots, n.
$$

Moreover, if

$$
d_j^{(m)} \leq z_j^{(m)}, \quad j = 1, 2, \ldots, i - 1
$$

and

$$
d_j^{(m-1)} \leq z_j^{(m-1)}, \quad j = i+1, \ldots, n
$$

holds then it follows from (20), (25) and (26) that

$$
\begin{aligned}
d_i^{(m)} &\leq \frac{\sum_{j=1}^{i-1}(|a_{ij}| + |a_{ij}^\tau|)d_j^{(m)} + \sum_{j=i+1}^{n}(|a_{ij}| + |a_{ij}^\tau|)d_j^{(m-1)}}{a_{ii} - 1 + |a_{ii}^\tau|} \\
&\leq \frac{\sum_{j=1}^{i-1}(|a_{ij}| + |a_{ij}^\tau|)z_j^{(m)} + \sum_{j=i+1}^{n}(|a_{ij}| + |a_{ij}^\tau|)z_j^{(m-1)}}{a_{ii} - 1 + |a_{ii}^\tau|} \\
&= z_i^{(m)}.
\end{aligned}
$$

Therefore $d_i^{(m)} \leq z_i^{(m)}$ holds for $i = 1, 2, \ldots, n$ and for all m. Since $\{z^{(m)}\}$ converges to the zero vector, so does $\{d^{(m)}\}$ and its convergence rate is faster than $\{z^{(m)}\}$. ∎

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