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Stable Patterns Realized by a Class of One-Dimensional Two-Layer CNNs

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***Abstract*— Stable patterns that can be realized by a class of one-dimensional two-layer cellular neural networks (CNNs) are studied in this paper. We first introduce the notions of potentially stable pattern, potentially stable local pattern, and local pattern set. We then show that all of 256 possible sets can be realized as the local pattern set of the two-layer CNN, while only 59 sets can be realized as the local pattern set of the single-layer CNN. We also propose a simple way to optimize the template values of the CNN, which is formulated as a set of linear programming problems, and present the obtained values for all of 256 sets.**

***Index Terms*— Cellular neural networks, stable patterns, two layers, hidden layer, template optimization**

I. Introduction

Cellular neural networks (CNNs) [1] are analog nonlinear circuits consisting of locally coupled cells. Global dynamical behavior of a CNN is completely determined by the network parameters represented by the feedback template, feedforward template and the bias. By choosing the values of these parameters appropriately, a CNN can perform various kinds of image processing tasks [2], [3]. However, as far as the standard CNN model [1] is concerned, there is a limitation on the signal processing capability due to their simple structure. To overcome this difficulty, various extensions of the model have been proposed so far, e.g., multilayer CNNs [1], [4]–[7], nonlinear templates [8], CNN universal machines [9], universal CNN cells [10], RTD-CNNs [11] and so on.

Among these models, the multilayer CNN not only is the most natural extension of, but also has a potential to perform more complex signal processing tasks than the original model. Several applications of multilayer CNNs in image processing have been proposed so far. Chua and Shi [4] proposed various multilayer CNNs for corner extraction from a noisy image, hole extraction, hole figure extraction, Radon transform, and so on. Wu *et al.* [5] showed that the Radon transform can be done by a two-layer CNN which is simpler than the three-layer CNN proposed in [4]. Yang *et al.* [7] have found many image processing tasks that can be carried out only by two-layer CNNs. In most of these applications, an input image is fed into a multilayer CNN as the initial state of or the input to a certain layer. Then the state of each cell evolves with time according to its state equation. After the state of the whole network converges to some steady state, the output of a specific layer is taken as the output image produced by the CNN. In this process, each layer plays different roles because coupling coefficients between cells are space-invariant in each layer but differ from layer to layer in general. This is the main advantage of multilayer CNNs.

How good is the signal processing capability of multilayer CNNs compared to single layer CNNs? This is an important question from both theoretical and practical points of view. As for single layer CNNs, there are some attempts to evaluate their signal processing capability. For example, Osuna and Moschytz [12] studied the separating capability of the standard CNN and gave upper and lower bounds for the number of different tasks that can be done by the standard CNN. Dogaru and Chua [10] proposed the universal CNN cell and investigated the number of Boolean functions that can be realized by the uncoupled CNN consisting of the universal cells. Chen *et al.* [13]–[15] have recently studied in detail the realization of Boolean functions by uncoupled standard CNNs. On the other hand, however, no attempt has been made so far for multilayer CNNs, to the best of the authors' knowledge. This is because the analysis of multilayer CNNs is much more difficult than single-layer CNNs due to the increase of the number of parameters.

The objective of this research is to characterize the signal processing capability of multilayer CNNs. As the first step toward the goal, we consider in this paper simple one-dimensional two-layer CNNs such that 1) there is no input to each layer (or all components of the feedforward templates are set to zero) , 2) the first layer is the output layer and the second layer works as the hidden layer, 3) each cell in the first layer receives output signals from three neighbors including itself in the first layer and from the nearest neighbor in the second layer, 4) each cell in the second layer receives output signals from itself and three neighbors in the first layer, and 5) self-coupling coefficients are greater than one. For this class of CNNs, we first introduce the notions of potentially stable pattern, potentially stable local pattern, and local pattern set. We then investigate how many local pattern sets can be realized by this class of CNNs. Since the set of all possible output patterns appearing in the first layer is

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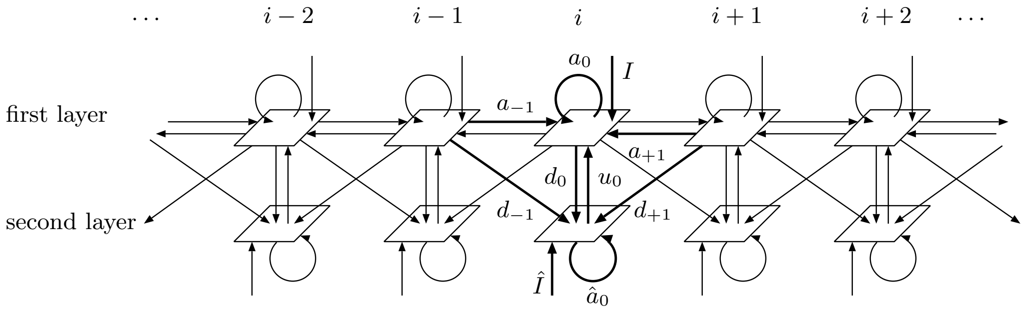

Fig. 1. Structure of one-dimensional two-layer CNNs described by (1).

completely determined by the local pattern set, the number of local pattern sets is a reasonable criterion to evaluate the signal processing capability of this class of CNNs. As a result, we show that all of 256 possible sets can be realized as the local pattern set while only 59 sets can be realized in the case of single-layer CNNs. This means that two-layer CNNs have a much higher potential for signal processing than single-layer CNNs. We finally propose a simple way to optimize the template values of the two-layer CNN, which is formulated as a set of linear programming problems, and present the obtained values for all of 256 sets.

II. Problem Formulation

A. CNN Model

Let us consider one-dimensional two-layer CNNs described by the system of differential equations:

$$\begin{cases} \dfrac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -x_i(t) + \displaystyle\sum_{j=-1}^{1} a_j y_{i+j}(t) + u_0 \hat{y}_i(t) + I \\ \dfrac{\mathrm{d}\hat{x}_i(t)}{\mathrm{d}t} = -\hat{x}_i(t) + \hat{a}_0 \hat{y}_i(t) + \displaystyle\sum_{j=-1}^{1} d_j y_{i+j}(t) + \hat{I}, \end{cases}$$
$$i = 1, 2, \dots, N \quad (1)$$

where N is the number of cells in each layer. $x_i(t)$ and $y_i(t)$ represent the state and output of the i-th cell in the first layer, respectively. The output $y_i(t)$ depends on the state $x_i(t)$ via the piecewise-linear function $f(\cdot)$ as follows:

$$y_i(t) = f(x_i(t)) \triangleq \frac{1}{2}(|x_i(t)+1| - |x_i(t)-1|).$$

Similarly, $\hat{x}_i(t)$ and $\hat{y}_i(t)$ represent the state and output of the i-th cell in the second layer, respectively, where the output $\hat{y}_i(t)$ depends on the state $\hat{x}_i(t)$ as

$$\hat{y}_i(t) = f(\hat{x}_i(t)) \triangleq \frac{1}{2}(|\hat{x}_i(t)+1| - |\hat{x}_i(t)-1|).$$

In the following, the state of the first layer, the output of the first layer, the state of the second layer, and the output of the second layer are denoted by $\boldsymbol{x}(t) = [x_1(t), x_2(t), \dots, x_N(t)]$, $\boldsymbol{y}(t) = [y_1(t), y_2(t), \dots, y_N(t)]$, $\hat{\boldsymbol{x}}(t) = [\hat{x}_1(t), \hat{x}_2(t), \dots, \hat{x}_N(t)]$ and $\hat{\boldsymbol{y}}(t) = [\hat{y}_1(t), \hat{y}_2(t), \dots, \hat{y}_N(t)]$, respectively[1]. Also, the state of the whole network is denoted by $\boldsymbol{s}(t) = [\boldsymbol{x}(t), \hat{\boldsymbol{x}}(t)] \in \mathbb{R}^{2N}$. Note that dynamical behavior of a CNN described by (1) is completely determined by the following parameters:

$$\boldsymbol{a} = [a_{-1}, a_0, a_{+1}],\ u_0,\ I,\ \hat{a}_0,\ \boldsymbol{d} = [d_{-1}, d_0, d_{+1}],\ \hat{I}$$

and the initial state $\boldsymbol{s}(0) = [\boldsymbol{x}(0), \hat{\boldsymbol{x}}(0)]$.

The structure of a CNN described by (1) is shown in Fig.1. The i-th cell in the first layer receives output signals from three neighboring cells including itself in the first layer and the i-th cell in the second layer, but neither from the $(i-1)$-th cell nor the $(i+1)$-th cell in the second layer. The i-th cell in the second layer receives output signals only from three neighboring cells in the first layer and itself, but not from its adjacent cells in the second layer.

Throughout this paper, we impose for simplicity the following two assumptions on (1).

Assumption 1: Boundary cells satisfy either the periodic boundary condition: $x_0(t) = x_N(t)$ and $x_{N+1}(t) = x_1(t)$ or the fixed boundary condition: $x_0(t) = x_{N+1}(t) = b \in \{1, -1\}$.

Assumption 2: Self-feedback coefficients a_0 and $\hat{a}_0$ are greater than one.

Note that we do not have to consider the boundary condition for the second layer. This is because each cell in the second layer does not receive output signals from the second layer except itself and thus boundary cells are not required.

It is well known that under Assumption 2 an equilibrium point $\boldsymbol{s}^* = [\boldsymbol{x}^*, \hat{\boldsymbol{x}}^*]$ is unstable unless $|x_i^*| > 1$ and $|\hat{x}_i^*| > 1$ hold for $i = 1, 2, \dots, N$ [16]. Conversely, it is easily seen that any equilibrium point $\boldsymbol{s}^* = [\boldsymbol{x}^*, \hat{\boldsymbol{x}}^*]$ such that $|x_i^*| > 1$ and $|\hat{x}_i^*| > 1$ for $i = 1, 2, \dots, N$ is stable. This implies that the output of each layer corresponding to any stable equilibrium point is necessarily a binary vector.

B. Potentially Stable Patterns and Potentially Stable Local Patterns

In this paper, the second layer of a CNN is supposed to work as the hidden layer. To be more specific, an input image (or pattern) $\boldsymbol{\alpha} \in [-1, 1]^N$ is first fed into the CNN as the initial

[1]Throughout this paper all vectors are assumed to be row vectors.

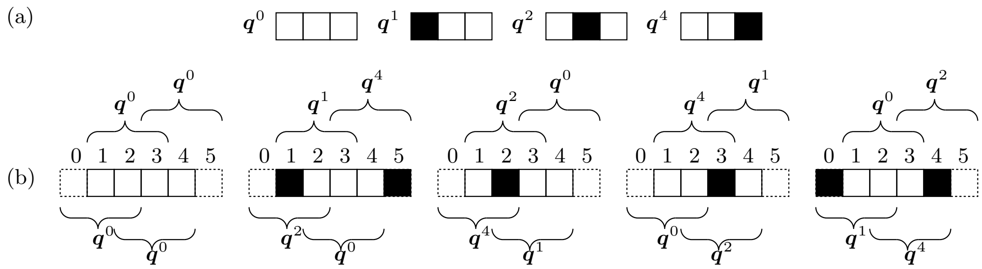

Fig. 2. Relationship between potentially stable local patterns and potentially stable patterns. (a) Potentially stable local patterns of a CNN with the parameters $\boldsymbol{a} = [-0.2, 1.3, -0.2]$, $u_0 = -1.0$, $I = 0.5$, $\hat{a}_0 = 3.0$, $\boldsymbol{d} = [-3, 1.1, -3]$ and $\hat{I} = 6.0$. (b) Potentially stable patterns of a 4-cell CNN with the boundary condition $x_0(t) = x_4(t)$ and $x_5(t) = x_1(t)$ for all t.

state of the first layer, that is, $\boldsymbol{x}(0) = \boldsymbol{\alpha}$, while the initial state of the second layer is set to a certain value independent of $\boldsymbol{\alpha}$, e.g. $\hat{\boldsymbol{x}}(0) = \boldsymbol{0}$; The state of each cell then evolves according to the state equation (1); If the state $\boldsymbol{s}(t) = [\boldsymbol{x}(t), \hat{\boldsymbol{x}}(t)]$ finally converges to an equilibrium point then the output of the first layer corresponding to the equilibrium point is regarded as the output image (or pattern) produced by the CNN.

Let us now give two definitions.

Definition 1 (Potentially Stable Pattern): An N-dimensional binary vector $\boldsymbol{p} = [p_1, p_2, \ldots, p_N] \in \{1, -1\}^N$ is said to be a potentially stable pattern of a CNN if there exists a stable equilibrium point $\boldsymbol{s}^* = [\boldsymbol{x}^*, \hat{\boldsymbol{x}}^*] \in \mathbb{R}^{2N}$ such that $f(x_i^*) = p_i$ for $i = 1, 2, \ldots, N$.

Definition 2 (Potentially Stable Local Pattern): A three-dimensional binary vector $\boldsymbol{q} = [q_{-1}, q_0, q_{+1}] \in \{1, -1\}^3$ is said to be a potentially stable local pattern of a CNN if the system of differential equations:

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} = & -x(t) + a_{-1}q_{-1} + a_0 f(x(t)) + a_{+1}q_{+1} \\ & +u_0 f(\hat{x}(t)) + I \\ \dfrac{\mathrm{d}\hat{x}(t)}{\mathrm{d}t} = & -\hat{x}(t) + \hat{a}_0 f(\hat{x}(t)) + \displaystyle\sum_{j=-1}^{1} d_j q_j + \hat{I} \end{cases}$$

has a stable equilibrium point $[x(t), \hat{x}(t)] = [x^*, \hat{x}^*] \in \mathbb{R}^2$ such that $f(x^*) = q_0$. The set of all potentially stable local patterns is called the local pattern set.

We hereafter express three-dimensional binary vectors as

$$\begin{array}{ll} \boldsymbol{q}^0 = [-1, -1, -1], & \boldsymbol{q}^1 = [+1, -1, -1], \\ \boldsymbol{q}^2 = [-1, +1, -1], & \boldsymbol{q}^3 = [+1, +1, -1], \\ \boldsymbol{q}^4 = [-1, -1, +1], & \boldsymbol{q}^5 = [+1, -1, +1], \\ \boldsymbol{q}^6 = [-1, +1, +1], & \boldsymbol{q}^7 = [+1, +1, +1] \end{array}$$

for the sake of simplicity.

It is obvious from Definitions 1 and 2 that under Assumption 1 a binary vector $\boldsymbol{p} = [p_1, p_2, \ldots, p_N] \in \{1, -1\}^N$ is a potentially stable pattern of a CNN if and only if $[p_{i-1}, p_i, p_{i+1}]$ is a potentially stable local pattern of the CNN for $i = 1, 2, \ldots, N$, where we assume $p_0 = p_N$ and $p_{N+1} = p_1$ for the periodic boundary condition and $p_0 = p_{N+1} = b \in \{1, -1\}$ for the fixed boundary condition. In this sense the set of potentially stable patterns is completely characterized by the local pattern set. For example, let us consider a CNN with the parameters $\boldsymbol{a} = [-0.2, 1.3, -0.2]$, $u_0 = -1.0$, $I = 0.5$, $\hat{a}_0 = 3.0$, $\boldsymbol{d} = [-3, 1.1, -3]$ and $\hat{I} = 6.0$. The local pattern set for this CNN is given by $\{\boldsymbol{q}^0, \boldsymbol{q}^1, \boldsymbol{q}^2, \boldsymbol{q}^4\}$. If we assume $N = 4$ and the periodic boundary condition, potentially stable patterns for this CNN are $[-1, -1, -1, -1]$, $[+1, -1, -1, -1]$, $[-1, +1, -1, -1]$, $[-1, -1, +1, -1]$ and $[-1, -1, -1, +1]$ (see Fig.2).

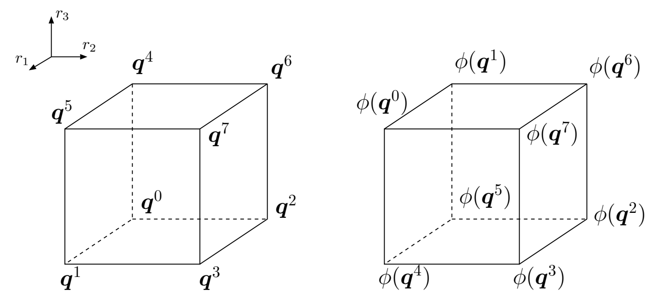

Fig. 3. Arrangements of $\{\boldsymbol{q}^i\}_{i=0}^7$ and $\{\phi(\boldsymbol{q}^i)\}_{i=0}^7$ in the space $\mathbb{R}^3 = \{[r_1, r_2, r_3] \,|\, r_i \in \mathbb{R},\, i = 1, 2, 3\}$.

The local pattern set of a CNN apparently takes one of 256 subsets of $\{1, -1\}^3$. The main purpose of this paper is to determine how many local pattern sets can be realized by the CNN described by (1).

In the following, the set $\{1, -1\}$ is denoted by $\mathbb{B}$. For any $\boldsymbol{q} = [q_{-1}, q_0, q_{+1}] \in \mathbb{B}^3$ we define the mapping $\phi(\boldsymbol{q})$ as follows:

$$\phi(\boldsymbol{q}) = [\phi_1(\boldsymbol{q}), \phi_2(\boldsymbol{q}), \phi_3(\boldsymbol{q})] = [q_0 q_{-1}, q_0, q_0 q_1]. \quad (2)$$

We also apply ϕ to any subset $S \subseteq \mathbb{B}^3$ as $\phi(S) = \{\phi(\boldsymbol{q}) \,|\, \boldsymbol{q} \in S\}$. The arrangements of $\{\boldsymbol{q}^i\}_{i=0}^7$ and $\{\phi(\boldsymbol{q}^i)\}_{i=0}^7$ in the space $\mathbb{R}^3 = \{[r_1, r_2, r_3] \,|\, r_i \in \mathbb{R},\, i = 1, 2, 3\}$ are shown in Fig.3. The Hamming distance between $\boldsymbol{q}^{i_1}$ and $\boldsymbol{q}^{i_2}$, which is defined by $\sum_{j=-1}^{1} |q_j^{i_1} - q_j^{i_2}|/2$, is denoted by $d_{\mathrm{H}}(\boldsymbol{q}^{i_1}, \boldsymbol{q}^{i_2})$. For example, $d_{\mathrm{H}}(\boldsymbol{q}^0, \boldsymbol{q}^2) = 1$. The difference between two sets A and B, which is defined as $\{x \,|\, x \in A \text{ and } x \notin B\}$, is denoted by $A \setminus B$. Two sets $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^n$ are said to be linearly separable if and only if there exist $\boldsymbol{w} \in \mathbb{R}^n$ and $b \in \mathbb{R}$ such that

$$\boldsymbol{w}\boldsymbol{x}^T + b \begin{cases} > 0, & \forall \boldsymbol{x} \in A \\ < 0, & \forall \boldsymbol{x} \in B \end{cases}.$$

III. Analysis of Local Pattern Sets

A. Single-Layer CNNs

Let us first consider the case where $u_0 = 0$. In this case, the state $\boldsymbol{x}(t)$ of the first layer is independent of the output $\hat{\boldsymbol{y}}(t)$ of the second layer, and thus the problem stated in the previous section corresponds to characterization of the family of local pattern sets that can be realized by single-layer CNNs.

Lemma 1: For a given set $S \subseteq \mathbb{B}^3$ there exists a CNN described by (1) with $u_0 = 0$ such that S is the local pattern set of the CNN if and only if $S_1 = \phi(S) \cup \{\mathbf{0}\}$ and $S_2 = \mathbb{B}^3 \setminus \phi(S)$ are linearly separable.

Proof: We will prove only necessity because sufficiency can be proved by reversing the following argument. Suppose S is the local pattern set of a CNN with $u_0 = 0$. Then for any vector $\boldsymbol{q} = [q_{-1}, q_0, q_{+1}] \in S$ there exists an x such that

$$-x + \sum_{j=-1}^{1} a_j q_j + I = 0 \quad \text{and} \quad x \begin{cases} > +1, & \text{if } q_0 = +1 \\ < -1, & \text{if } q_0 = -1 \end{cases}$$

which is equivalent to the inequality

$$q_0(a_{-1}q_{-1} + a_0 q_0 + a_{+1}q_{+1} + I) > 1\,.$$

From this inequality we have

$$[a_{-1}, I, a_{+1}][q_0 q_{-1}, q_0, q_0 q_{+1}]^T + a_0 - 1 > 0, \qquad \forall \boldsymbol{q} = [q_{-1}, q_0, q_{+1}] \in S\,. \tag{3}$$

On the other hand, since the inequality

$$q_0(a_{-1}q_{-1} + a_0 q_0 + a_{+1}q_{+1} + I) \le 1$$

holds for any vector $\boldsymbol{q} \in \mathbb{B}^3 \setminus S$, we have

$$[a_{-1}, I, a_{+1}][q_0 q_{-1}, q_0, q_0 q_{+1}]^T + a_0 - 1 \le 0, \qquad \forall \boldsymbol{q} = [q_{-1}, q_0, q_{+1}] \in \mathbb{B}^3 \setminus S\,. \tag{4}$$

It follows from Eqs.(3) and (4) that $\phi(S)$ and $\mathbb{B}^3 \setminus \phi(S)$ are linearly separable[2]. Moreover, since $a_0 > 1$, we have

$$[a_{-1}, I, a_{+1}][0, 0, 0]^T + a_0 - 1 > 0.$$

Therefore we can conclude that $S_1 = \phi(S) \cup \{\mathbf{0}\}$ and $S_2 = \mathbb{B}^3 \setminus \phi(S)$ are linearly separable. ■

By using this lemma, we can derive the following theorem.

Theorem 1: The number of subsets of $\mathbb{B}^3$ that can be realized as the local pattern set of a CNN in the form of (1) with $u_0 = 0$ is 59.

Proof: In order for $S \subseteq \mathbb{B}^3$ to be the local pattern set of a CNN, at least one of two vectors $\boldsymbol{q}^i$ and $\boldsymbol{q}^j$ such that $\phi(\boldsymbol{q}^i) = -\phi(\boldsymbol{q}^j)$ must belong to S because otherwise $\phi(S) \cup \{\mathbf{0}\}$ and $\mathbb{B}^3 \setminus S$ are not linearly separable. Thus $|S|$, the number of elements of S, must be at least 4. In the following, we will study the condition for S to be the local pattern set for each value of $|S|$.

1) Suppose $|S| = 8$ (i.e., $S = \mathbb{B}^3$). In this case, we can realize S as the local pattern set by setting a_0 to any value greater than one and $a_{-1} = a_1 = I = 0$.

[2] The equal sign in (4) can be removed without affecting (3) by decreasing a_0 slightly.

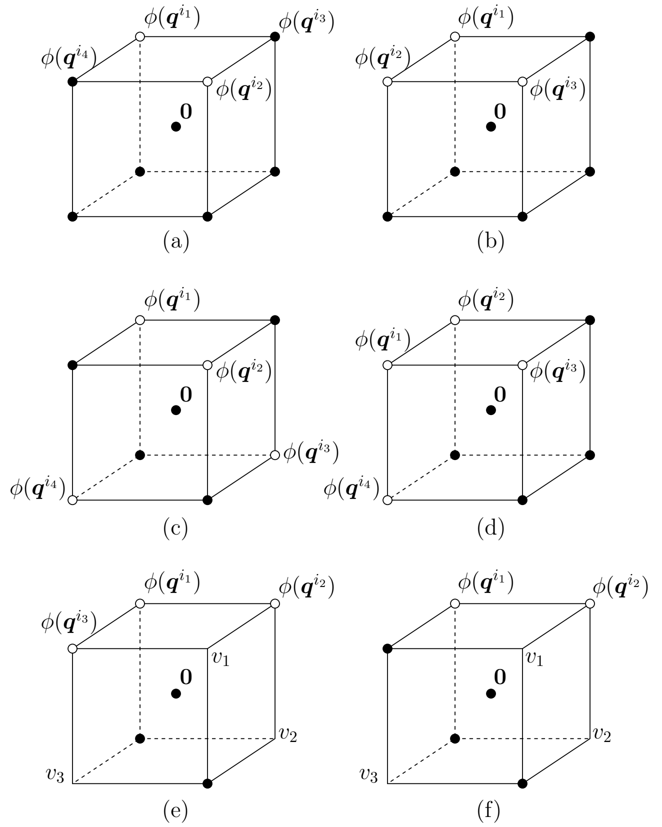

Fig. 4. Various cases considered in the proof of Theorem 1. A black (white, resp.) circle indicates that the three-dimensional binary vector corresponds to the vertex belongs to S ($\mathbb{B}^3 \setminus S$, resp.).

2) Suppose $|S| = 7$. It is obvious that $\phi(S) \cup \{\mathbf{0}\}$ and $\mathbb{B}^3 \setminus \phi(S)$ are linearly separable. Thus S can be realized as the local pattern set. The number of different S such that $|S| = 7$ is apparently 8.
3) Suppose $|S| = 6$ and $\mathbb{B}^3 \setminus S = \{\boldsymbol{q}^{i_1}, \boldsymbol{q}^{i_2}\}$. There are three cases according to the value of $d_{\mathrm{H}}(\phi(\boldsymbol{q}^{i_1}), \phi(\boldsymbol{q}^{i_2}))$.
   a) $d_{\mathrm{H}}(\phi(\boldsymbol{q}^{i_1}), \phi(\boldsymbol{q}^{i_2})) = 1$. It is obvious that $\phi(S) \cup \{\mathbf{0}\}$ and $\mathbb{B}^3 \setminus \phi(S)$ are linearly separable. Thus S can be realized as the local pattern set. The number of different sets $\{\boldsymbol{q}^{i_1}, \boldsymbol{q}^{i_2}\}$ such that $d_{\mathrm{H}}(\phi(\boldsymbol{q}^{i_1}), \phi(\boldsymbol{q}^{i_2})) = 1$ is 12, which is equal to the number of edges of the cube.
   b) $d_{\mathrm{H}}(\phi(\boldsymbol{q}^{i_1}), \phi(\boldsymbol{q}^{i_2})) = 2$. There are two patterns $\boldsymbol{q}^{i_3}$ and $\boldsymbol{q}^{i_4}$ such that $\{\boldsymbol{q}^{i_3}, \boldsymbol{q}^{i_4}\} \subset S$ and $\phi_j(\boldsymbol{q}^{i_1}) = \phi_j(\boldsymbol{q}^{i_2}) = \phi_j(\boldsymbol{q}^{i_3}) = \phi_j(\boldsymbol{q}^{i_4}) =$ for some $j \in \{-1, 0, 1\}$ as shown in Fig.4(a). Since two sets $\phi(\{\boldsymbol{q}^{i_1}, \boldsymbol{q}^{i_2}\})$ and $\phi(\{\boldsymbol{q}^{i_3}, \boldsymbol{q}^{i_4}\}) \subset \phi(S)$ are not linearly separable, S cannot be realized as the local pattern set.
   c) $d_{\mathrm{H}}(\phi(\boldsymbol{q}^{i_1}), \phi(\boldsymbol{q}^{i_2})) = 3$. Since $\phi(\boldsymbol{q}^{i_1}) = -\phi(\boldsymbol{q}^{i_2})$ holds in this case, two sets $\phi(\{\boldsymbol{q}^{i_1}, \boldsymbol{q}^{i_2}\})$ and $\{\mathbf{0}\}$ are not linearly separable. Therefore S cannot be realized as the local pattern set.
4) Suppose $|S| = 5$ and $\mathbb{B}^3 \setminus S = \{\boldsymbol{q}^{i_1}, \boldsymbol{q}^{i_2}, \boldsymbol{q}^{i_3}\}$. We assume without loss of generality that $d_{\mathrm{H}}(\phi(\boldsymbol{q}^{i_1}),$

$\phi(\boldsymbol{q}^{i_2})) \leq d_{\mathrm{H}}(\phi(\boldsymbol{q}^{i_2}), \phi(\boldsymbol{q}^{i_3})) \leq d_{\mathrm{H}}(\phi(\boldsymbol{q}^{i_3}), \phi(\boldsymbol{q}^{i_1}))$. If $d_{\mathrm{H}}(\phi(\boldsymbol{q}^{i_3}), \phi(\boldsymbol{q}^{i_1})) = 3$ holds, S cannot be realized as the local pattern set for the same reason as Case 3-c. Also, if $d_{\mathrm{H}}(\phi(\boldsymbol{q}^{i_1}), \phi(\boldsymbol{q}^{i_2})) = d_{\mathrm{H}}(\phi(\boldsymbol{q}^{i_2}), \phi(\boldsymbol{q}^{i_3})) = d_{\mathrm{H}}(\phi(\boldsymbol{q}^{i_3}), \phi(\boldsymbol{q}^{i_1})) = 2$ holds, $\phi(\{\boldsymbol{q}^{i_1}, \boldsymbol{q}^{i_2}, \boldsymbol{q}^{i_3}\})$ and $\{\mathbf{0}\}$ are not linearly separable because $\mathbf{0}$ is the center of the triangle whose vertices are $\phi(\boldsymbol{q}^{i_1})$, $\phi(\boldsymbol{q}^{i_2})$ and $\phi(\boldsymbol{q}^{i_3})$. Hence S cannot be realized as the local pattern set in this case. So we can concentrate our attention on the case where $d_{\mathrm{H}}(\phi(\boldsymbol{q}^{i_1}), \phi(\boldsymbol{q}^{i_2})) = 1$ and $d_{\mathrm{H}}(\phi(\boldsymbol{q}^{i_3}), \phi(\boldsymbol{q}^{i_1})) = 2$. Furthermore, we easily see that $d_{\mathrm{H}}(\phi(\boldsymbol{q}^{i_2}), \phi(\boldsymbol{q}^{i_3}))$ must be 1 in this case. Since $\phi(\boldsymbol{q}^{i_1})$, $\phi(\boldsymbol{q}^{i_2})$ and $\phi(\boldsymbol{q}^{i_3})$ are on the same face of the cube as shown in Fig.4(b), it is obvious that $\phi(S) \cup \{\mathbf{0}\}$ and $\mathbb{B}^3 \setminus \phi(S)$ are linearly separable. Thus S can be realized as the local pattern set. The number of different sets $\{\boldsymbol{q}^{i_1}, \boldsymbol{q}^{i_2}, \boldsymbol{q}^{i_3}\}$ satisfying $d_{\mathrm{H}}(\phi(\boldsymbol{q}^{i_1}), \phi(\boldsymbol{q}^{i_2})) = d_{\mathrm{H}}(\phi(\boldsymbol{q}^{i_2}), \phi(\boldsymbol{q}^{i_3})) = 1$ and $d_{\mathrm{H}}(\phi(\boldsymbol{q}^{i_3}), \phi(\boldsymbol{q}^{i_1})) = 2$ is 24, which is equal to the number of faces of the cube times the number of vertices on a face.

5) Suppose $|S| = 4$ and $\mathbb{B}^3 \setminus S = \{\boldsymbol{q}^{i_1}, \boldsymbol{q}^{i_2}, \boldsymbol{q}^{i_3}, \boldsymbol{q}^{i_4}\}$. We assume without loss of generality that $d_{\mathrm{H}}(\phi(\boldsymbol{q}^{i_1}), \phi(\boldsymbol{q}^{i_2})) \leq d_{\mathrm{H}}(\phi(\boldsymbol{q}^{i_1}), \phi(\boldsymbol{q}^{i_3})) \leq d_{\mathrm{H}}(\phi(\boldsymbol{q}^{i_1}), \phi(\boldsymbol{q}^{i_4}))$. If $d_{\mathrm{H}}(\phi(\boldsymbol{q}^{i_1}), \phi(\boldsymbol{q}^{i_4})) = 3$ holds, S cannot be realized as the local pattern set for the same reason as Case 3-c. Also, if $d_{\mathrm{H}}(\phi(\boldsymbol{q}^{i_1}), \phi(\boldsymbol{q}^{i_j})) = 2$ holds for $j = 2, 3$ and 4, S cannot be realized as the local pattern set because the arrangement of $\phi(\{\boldsymbol{q}^{i_1}, \boldsymbol{q}^{i_2}, \boldsymbol{q}^{i_3}, \boldsymbol{q}^{i_4}\})$ is as shown in Fig.4(c). From these observations, we can concentrate our attention on the following three cases.

a) $d_{\mathrm{H}}(\phi(\boldsymbol{q}^{i_1}), \phi(\boldsymbol{q}^{i_j})) = 1$ for $j = 2, 3$ and 4. In this case, $\phi(S) \cup \{\mathbf{0}\}$ and $\mathbb{B}^3 \setminus \phi(S)$ are linearly separable as shown in Fig.4(d), and thus S can be realized as the local pattern set. The number of different sets $\{\boldsymbol{q}^{i_1}, \boldsymbol{q}^{i_2}, \boldsymbol{q}^{i_3}, \boldsymbol{q}^{i_4}\}$ corresponding to this case is 8, which is equal to the number of vertices of the cube.

b) $d_{\mathrm{H}}(\phi(\boldsymbol{q}^{i_1}), \phi(\boldsymbol{q}^{i_2})) = d_{\mathrm{H}}(\phi(\boldsymbol{q}^{i_1}), \phi(\boldsymbol{q}^{i_3})) = 1$ and $d_{\mathrm{H}}(\phi(\boldsymbol{q}^{i_1}), \phi(\boldsymbol{q}^{i_4})) = 2$. We assume without loss of generality that the arrangement of $\phi(\{\boldsymbol{q}^{i_1}, \boldsymbol{q}^{i_2}, \boldsymbol{q}^{i_3}\})$ is as shown in Fig.4(e). If $\phi(\boldsymbol{q}^{i_4})$ is at v_2 or v_3, S cannot be realized as the local pattern set because $d_{\mathrm{H}}(\phi(\boldsymbol{q}^{i_2}), \phi(\boldsymbol{q}^{i_4})) = 3$ or $d_{\mathrm{H}}(\phi(\boldsymbol{q}^{i_3}), \phi(\boldsymbol{q}^{i_4})) = 3$ holds, respectively. On the other hand, if $\phi(\boldsymbol{q}^{i_4})$ is at v_1 then $\phi(\boldsymbol{q}^{i_1})$, $\phi(\boldsymbol{q}^{i_2})$, $\phi(\boldsymbol{q}^{i_3})$ and $\phi(\boldsymbol{q}^{i_4})$ are on the same face of the cube and thus S can be realized as the local pattern set. The number of different sets $\{\boldsymbol{q}^{i_1}, \boldsymbol{q}^{i_2}, \boldsymbol{q}^{i_3}, \boldsymbol{q}^{i_4}\}$ corresponding to this case is 6, which is equal to the number of faces of the cube.

c) $d_{\mathrm{H}}(\phi(\boldsymbol{q}^{i_1}), \phi(\boldsymbol{q}^{i_2})) = 1$ and $d_{\mathrm{H}}(\phi(\boldsymbol{q}^{i_1}), \phi(\boldsymbol{q}^{i_3})) = d_{\mathrm{H}}(\phi(\boldsymbol{q}^{i_1}), \phi(\boldsymbol{q}^{i_4})) = 2$. We assume without loss of generality that the arrangement of $\phi(\{\boldsymbol{q}^{i_1}, \boldsymbol{q}^{i_2}\})$ is as shown in Fig.4(f). If either $\phi(\boldsymbol{q}^{i_3})$ or $\phi(\boldsymbol{q}^{i_4})$ is at v_3, S cannot be realized as the local pattern set because $d_{\mathrm{H}}(\phi(\boldsymbol{q}^{i_2}), \phi(\boldsymbol{q}^{i_3})) = 3$ or $d_{\mathrm{H}}(\phi(\boldsymbol{q}^{i_2}), \phi(\boldsymbol{q}^{i_4})) = 3$ holds, respectively. Conversely, if neither $\phi(\boldsymbol{q}^{i_3})$ nor $\phi(\boldsymbol{q}^{i_4})$ is at v_3, $d_{\mathrm{H}}(\phi(\boldsymbol{q}^{i_2}), \phi(\boldsymbol{q}^{i_j})) = 1$ holds for $j = 1, 3$ and 4. This case has already been considered in Case 5-a.

Total number of S that can be realized as the local pattern set is $1 + 8 + 12 + 24 + (8 + 6) = 59$. This completes the proof. ∎

B. Two-Layer CNNs

Let us next consider the general case where $u_0 \neq 0$. The following theorem is the main result of this paper.

Theorem 2: All of 256 subsets of $\mathbb{B}^3$ can be realized as the local pattern set of a CNN described by (1).

Comparing Theorem 2 with Theorem 1, we can conclude that two-layer CNNs have a much higher potential for signal processing than single-layer CNNs.

The remainder of this subsection is devoted to the proof of Theorem 2 which consists of seven lemmas. We only consider $S \subseteq \mathbb{B}^3$ such that $|S| \leq 6$ because if $|S| \geq 7$ then S can be realized as the local pattern set of a single layer CNN. First we present Lemma 2 which gives us useful information on how to determine the values of the parameters $\hat{a}_0$, $\boldsymbol{d} = [d_{-1}, d_0, d_{+1}]$ and $\hat{I}$. Next we present two lemmas (Lemmas 3 and 4) which play key roles in the proof of Theorem 2. In fact, the realizability of S as the local pattern set can be proved by using Lemmas 3 and 4 in most cases, as shown in Lemma 5. We finally present three lemmas (Lemmas 6–8) in order to deal with the remaining special cases.

It follows from Definition 2 that a binary vector $\boldsymbol{q} \in \mathbb{B}^3$ is a potentially stable local pattern of a CNN if and only if

$$[a_{-1}, I, a_{+1}]\, \phi(\boldsymbol{q})^T + a_0 + q_0 u_0 f(\hat{x}^*) > 1 \tag{5}$$

holds, where $\hat{x}^*$ is any stable equilibrium point of the following differential equation:

$$\frac{\mathrm{d}\hat{x}(t)}{\mathrm{d}t} = -\hat{x}(t) + \hat{a}_0 f(\hat{x}(t)) + \sum_{j=-1}^{1} d_j q_j + \hat{I} \tag{6}$$

Lemma 2: If $\hat{a}_0 - 1 < |\sum_{j=-1}^{1} d_j q_j + \hat{I}|$ then (6) has a unique equilibrium point $\hat{x}^*$ which is stable. Moreover $f(\hat{x}^*)$ is given by the following equation:

$$f(\hat{x}^*) = \begin{cases} +1, & \text{if } \sum_{j=-1}^{1} d_j q_j + \hat{I} > 0 \\ -1, & \text{if } \sum_{j=-1}^{1} d_j q_j + \hat{I} < 0 \end{cases}$$

Proof: Let us denote $\sum_{j=-1}^{1} d_j q_j + \hat{I}$ by $\hat{I}'$ for simplicity. Eq.(6) has an equilibrium point $\hat{x}^* > 1$ if and only if $-\hat{x} + \hat{a}_0 + \hat{I}' = 0$ has a solution in the interval $(1, \infty)$ which is equivalent to

$$\hat{a}_0 - 1 > -\hat{I}'\,. \tag{7}$$

Similarly, Eq.(6) has an equilibrium point $\hat{x}^* < -1$ if and only if $-\hat{x} - \hat{a}_0 + \hat{I}' = 0$ has a solution in the interval $(-\infty, -1)$ which is equivalent to

$$\hat{a}_0 - 1 > \hat{I}'\,. \tag{8}$$

On the other hand, Eq.(6) has an equilibrium point $\hat{x}^*$ such that $|\hat{x}^*| \leq 1$ if and only if $-\hat{x} + \hat{a}_0\hat{x} + \hat{I}' = 0$ has a solution in the interval $[-1, 1]$ which is equivalent to

$$\hat{a}_0 - 1 \geq |\hat{I}'|\,. \tag{9}$$

Note that $0 < \hat{a}_0 - 1 < |\hat{I}'|$ implies $\hat{I}' \neq 0$. If $\hat{I}' > 0$ then (7) holds while (8) and (9) do not hold. Therefore (6) has a unique equilibrium point $\hat{x}^* = \hat{a}_0 + \hat{I}' > 1$ which is stable. If $\hat{I}' < 0$ then (8) holds while (7) and (9) do not hold. Therefore (6) has a unique equilibrium point $\hat{x}^* = -\hat{a}_0 + \hat{I}' < -1$ which is stable. ∎

Lemma 3: Let S_1 be a subset of $\mathbb{B}^3$ such that $\phi(S_1) \cup \{\mathbf{0}\}$ and $\mathbb{B}^3 \setminus \phi(S_1)$ are linearly separable. Let S_2 be a subset of S_1 such that i) either $S_2 \subseteq \{\boldsymbol{q} \in \mathbb{B}^3 \,|\, q_0 = +1\}$ or $S_2 \subseteq \{\boldsymbol{q} \in \mathbb{B}^3 \,|\, q_0 = -1\}$ holds, and ii) S_2 and $\mathbb{B}^3 \setminus S_2$ are linearly separable. Then $S_1 \setminus S_2$ can be realized as the local pattern set of a CNN described by (1).

Proof: Since $\phi(S_1) \cup \{\mathbf{0}\}$ and $\mathbb{B}^3 \setminus \phi(S_1)$ are linearly separable, there exist parameters $w_0\,(> 1)$, w_1, w_2, and w_3 such that

$$[w_1, w_2, w_3]\,\phi(\boldsymbol{q})^T + w_0 \begin{cases} > 1, & \text{if } \boldsymbol{q} \in S_1 \\ < 1, & \text{if } \boldsymbol{q} \notin S_1 \end{cases} \tag{10}$$

holds. Since S_2 and $\mathbb{B}^3 \setminus S_2$ are linearly separable, it follows from Lemma 2 that there exists a set of parameters $\boldsymbol{d} = [d_{-1}, d_0, d_{+1}]$, $\hat{a}_0$ and $\hat{I}$ such that (6) has a unique equilibrium point $\hat{x}^*$ for each $\boldsymbol{q} \in \mathbb{B}^3$ and $f(\hat{x}^*)$ is given by

$$f(\hat{x}^*) = \begin{cases} 1, & \text{if } \boldsymbol{q} \in S_2 \\ -1, & \text{if } \boldsymbol{q} \notin S_2 \end{cases} \tag{11}$$

Let $a_{-1} = w_1$, $a_0 = w_0$, $a_{+1} = w_3$, $u_0 = L$ and $I = w_2 + L$ where L is a sufficiently small negative number if $S_2 \subseteq \{\boldsymbol{q} \in \mathbb{B}^3 \,|\, q_0 = 1\}$ and a sufficiently large positive number if $S_2 \subseteq \{\boldsymbol{q} \in \mathbb{B}^3 \,|\, q_0 = -1\}$. Then we have

$$\begin{aligned} &[a_{-1}, I, a_{+1}]\,\phi(\boldsymbol{q})^T + a_0 + q_0 u_0 f(\hat{x}^*) \\ &\quad = \quad [w_1, w_2 + L, w_3]\,\phi(\boldsymbol{q})^T + w_0 + q_0 L f(\hat{x}^*) \\ &\quad = \quad [w_1, w_2, w_3]\,\phi(\boldsymbol{q})^T + w_0 + q_0 L(f(\hat{x}^*) + 1). \end{aligned} \tag{12}$$

It follows from (11) and the definition of L that

$$q_0 L(f(\hat{x}^*) + 1) = \begin{cases} -2|L|, & \text{if } \boldsymbol{q} \in S_2 \\ 0, & \text{if } \boldsymbol{q} \notin S_2 \end{cases} \tag{13}$$

Since $|L|$ is sufficiently large, we have from (10), (12) and (13)

$$\begin{aligned} &[a_{-1}, I, a_{+1}]\,\phi(\boldsymbol{q})^T + a_0 + q_0 u_0 f(\hat{x}^*) \\ &\qquad\qquad \begin{cases} > 1, & \text{if } \boldsymbol{q} \in S_1 \setminus S_2 \\ < 1, & \text{if } \boldsymbol{q} \notin S_1 \setminus S_2 \end{cases} \end{aligned}$$

which means $S_1 \setminus S_2$ is the local pattern set of the CNN. ∎

In a similar way, we can derive the following lemma.

Lemma 4: Let S_1 be a subset of $\mathbb{B}^3$ such that $\phi(S_1) \cup \{\mathbf{0}\}$ and $\mathbb{B}^3 \setminus \phi(S_1)$ are linearly separable. Let S_2 be a subset of $\mathbb{B}^3 \setminus S_1$ such that i) either $S_2 \subseteq \{\boldsymbol{q} \in \mathbb{B}^3 \,|\, q_0 = +1\}$ or $S_2 \subseteq \{\boldsymbol{q} \in \mathbb{B}^3 \,|\, q_0 = -1\}$ holds, and ii) S_2 and $\mathbb{B}^3 \setminus S_2$ are linearly separable. Then $S_1 \cup S_2$ can be realized as the local pattern set of a CNN described by (1).

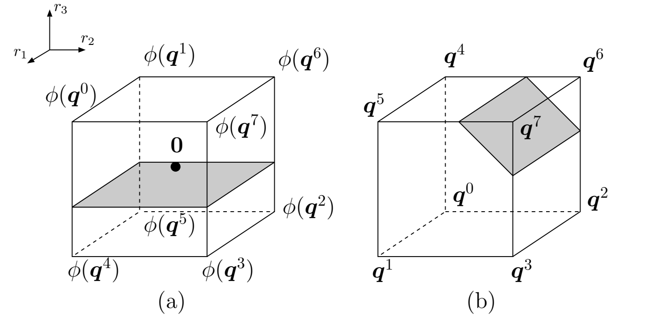

Fig. 5. How to realize $\{\boldsymbol{q}^0, \boldsymbol{q}^1\}$ as the local pattern set by using Lemma 3. (a) Linear separation of $\phi(S_1) \cup \{\mathbf{0}\}$ and $\mathbb{B}^3 \setminus \phi(S_1)$. (b) Linear separation of S_2 and $\mathbb{B}^3 \setminus S_2$.

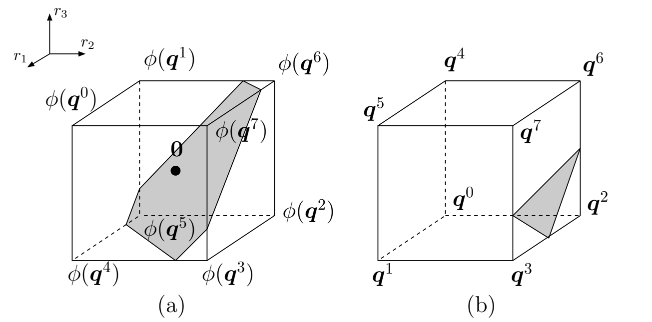

Fig. 6. How to realize $\{\boldsymbol{q}^0, \boldsymbol{q}^1, \boldsymbol{q}^2, \boldsymbol{q}^4, \boldsymbol{q}^7\}$ as the local pattern set by using Lemma 4. (a) Linear separation of $\phi(S_1) \cup \{\mathbf{0}\}$ and $\mathbb{B}^3 \setminus \phi(S_1)$. (b) Linear separation of S_2 and $\mathbb{B}^3 \setminus S_2$.

Fig. 5 shows how $S = \{\boldsymbol{q}^0, \boldsymbol{q}^1\}$ is realized as the local pattern set of a CNN by using Lemma 3. Let $S_1 = \{\boldsymbol{q}^0, \boldsymbol{q}^1, \boldsymbol{q}^6, \boldsymbol{q}^7\}$ and $S_2 = \{\boldsymbol{q}^6, \boldsymbol{q}^7\}$. Then S_2 is a subset of both S_1 and $\{\boldsymbol{q} \in \mathbb{B}^3 \,|\, q_0 = +1\}$. As shown in Fig. 5(a), $\phi(S_1) \cup \{\mathbf{0}\}$ and $\mathbb{B}^3 \setminus \phi(S_1)$ are linearly separable with the plane indicated by gray. Also, as shown in Fig. 5(b), S_2 and $\mathbb{B}^3 \setminus S_2$ are linearly separable with the plane indicated by gray. Therefore, according to Lemma 3, $S_1 \setminus S_2 = S$ can be realized as the local pattern set.

Fig. 6 shows how $S = \{\boldsymbol{q}^0, \boldsymbol{q}^1, \boldsymbol{q}^2, \boldsymbol{q}^4, \boldsymbol{q}^7\}$ is realized as the local pattern set of a CNN by using Lemma 4. Let $S_1 = \{\boldsymbol{q}^0, \boldsymbol{q}^1, \boldsymbol{q}^4, \boldsymbol{q}^7\}$ and $S_2 = \{\boldsymbol{q}^2\}$. Then S_2 is a subset of $\{\boldsymbol{q} \in \mathbb{B}^3 \,|\, q_0 = +1\}$. As shown in Fig. 6(a), $\phi(S_1) \cup \{\mathbf{0}\}$ and $\mathbb{B}^3 \setminus \phi(S_1)$ are linearly separable with the plane indicated by gray. Also, as shown in Fig. 6(b), S_2 and $\mathbb{B}^3 \setminus S_2$ are linearly separable with the plane indicated by gray. Therefore, according to Lemma 4, $S_1 \cup S_2 = S$ can be realized as the local pattern set.

The following lemma follows from Lemmas 1, 3 and 4.

Lemma 5: If $S \subset \mathbb{B}^3$ satisfies one of the following conditions, S can be realized as the local pattern set of a CNN described by (1).

1) $S = \emptyset$.
2) $|S| = 1$.
3) $S = \{\boldsymbol{q}^{i_1}, \boldsymbol{q}^{i_2}\}$ where one of the following conditions holds.

a) $q_0^{i_1} \neq q_0^{i_2}$
b) $q_0^{i_1} = q_0^{i_2}$, $d_{\mathrm{H}}(\boldsymbol{q}^{i_1}, \boldsymbol{q}^{i_2}) = 1$

4) $S = \{\boldsymbol{q}^{i_1}, \boldsymbol{q}^{i_2}, \boldsymbol{q}^{i_3}\}$ where one of the following conditions holds.
   a) $q_0^{i_1} = q_0^{i_2} = q_0^{i_3}$
   b) $q_0^{i_1} = q_0^{i_2} \neq q_0^{i_3}$, $d_{\mathrm{H}}(\boldsymbol{q}^{i_1}, \boldsymbol{q}^{i_2}) = 1$
   c) $d_{\mathrm{H}}(\boldsymbol{q}^{i_1}, \boldsymbol{q}^{i_2}) = d_{\mathrm{H}}(\boldsymbol{q}^{i_2}, \boldsymbol{q}^{i_3}) = d_{\mathrm{H}}(\boldsymbol{q}^{i_3}, \boldsymbol{q}^{i_1}) = 2$
5) $S = \{\boldsymbol{q}^{i_1}, \boldsymbol{q}^{i_2}, \boldsymbol{q}^{i_3}, \boldsymbol{q}^{i_4}\} \subset \mathbb{B}^3$ where one of the following conditions holds.
   a) $q_0^{i_1} = q_0^{i_2} = q_0^{i_3} = q_0^{i_4}$
   b) $q_0^{i_1} = q_0^{i_2} = q_0^{i_3} \neq q_0^{i_4}$
   c) $q_0^{i_1} = q_0^{i_2} \neq q_0^{i_3} = q_0^{i_4}$, $d_{\mathrm{H}}(\boldsymbol{q}^{i_1}, \boldsymbol{q}^{i_2}) = d_{\mathrm{H}}(\boldsymbol{q}^{i_3}, \boldsymbol{q}^{i_4}) = 1$
   d) $q_0^{i_1} = q_0^{i_2} \neq q_0^{i_3} = q_0^{i_4}$, $d_{\mathrm{H}}(\boldsymbol{q}^{i_1}, \boldsymbol{q}^{i_2}) = 1$, $d_{\mathrm{H}}(\boldsymbol{q}^{i_3}, \boldsymbol{q}^{i_4}) = 2$
6) $|S| = 5$.
7) $|S| = 6$.

Proof: For each case the claim can be proved as follows.

1) Let $S_1 = S_2 = \{\boldsymbol{q}^2, \boldsymbol{q}^3, \boldsymbol{q}^6, \boldsymbol{q}^7\}$. Since S_1 and S_2 satisfy the conditions in Lemma 3, $S_1 \setminus S_2 = \emptyset$ can be realized as the local pattern set.
2) Let $S = \{\boldsymbol{q}^i\}$. If we set $S_1 = \{\boldsymbol{q} \in \mathbb{B}^3 \,|\, q_0 = q_0^i\}$ and $S_2 = S_1 \setminus S$ then S_1 and S_2 satisfy the conditions in Lemma 3. Therefore $S_1 \setminus S_2 = S$ can be realized as the local pattern set.
3) We set S_1 and S_2 as follows. It is easily seen for all cases that S_1 and S_2 satisfy the conditions in Lemma 3 and that $S_1 \setminus S_2 = S$. Hence S can be realized as the local pattern set.
   a) $S_1 = \{\boldsymbol{q} \in \mathbb{B}^3 \,|\, q_0 = q_0^{i_1}\} \cup \{\boldsymbol{q}^{i_2}\}$, $S_2 = S_1 \setminus S$
   b) $S_1 = \{\boldsymbol{q} \in \mathbb{B}^3 \,|\, q_0 = q_0^{i_1}\}$, $S_2 = S_1 \setminus S$
4) We set S_1 and S_2 as follows. It is easily seen for all cases that S_1 and S_2 satisfy the conditions in Lemma 3 and that $S_1 \setminus S_2 = S$. Hence S can be realized as the local pattern set.
   a) $S_1 = \{\boldsymbol{q} \in \mathbb{B}^3 \,|\, q_0 = q_0^{i_1}\}$, $S_2 = S_1 \setminus S$
   b) $S_1 = \{\boldsymbol{q} \in \mathbb{B}^3 \,|\, q_0 = q_0^{i_1}\} \cup \{\boldsymbol{q}^{i_3}\}$, $S_2 = S_1 \setminus S$
   c) $S_1 = S \cup \{\boldsymbol{q}^*\}$, $S_2 = \{\boldsymbol{q}^*\}$ where $\boldsymbol{q}^* \in \mathbb{B}^3$ such that $d_{\mathrm{H}}(\phi(\boldsymbol{q}^*), \phi(\boldsymbol{q}^{i_1})) = d_{\mathrm{H}}(\phi(\boldsymbol{q}^*), \phi(\boldsymbol{q}^{i_2})) = d_{\mathrm{H}}(\phi(\boldsymbol{q}^*), \phi(\boldsymbol{q}^{i_3})) = 1$
5) a) Since $\phi(S_1) = \{\boldsymbol{q} \in \mathbb{B}^3 \,|\, q_0 = q_0^{i_1}\}$ holds, $\phi(S_1) \cup \{\boldsymbol{0}\}$ and $\mathbb{B}^3 \setminus \phi(S_1)$ are linearly separable. Hence it follows from Lemma 1 that S can be realized by a CNN with $u_0 = 0$.
   b) Let $S_1 = \{\boldsymbol{q} \in \mathbb{B}^3 \,|\, q_0 = q_0^{i_1}\} \cup \{\boldsymbol{q}^{i_4}\}$ and $S_2 = S_1 \setminus S$. Then S_1 and S_2 satisfy the conditions in Lemma 3 and hence $S_1 \setminus S_2 = S$ can be realized as the local pattern set.
   c) Let $S_1 = \{\boldsymbol{q} \in \mathbb{B}^3 \,|\, q_0 = q_0^{i_1}\} \cup \{\boldsymbol{q}^{i_3}, \boldsymbol{q}^{i_4}\}$ and $S_2 = S_1 \setminus S$. Since $d_{\mathrm{H}}(\phi(\boldsymbol{q}^{i_3}), \phi(\boldsymbol{q}^{i_4})) = 1$, it is apparent that $\phi(S_1) \cup \{\boldsymbol{0}\}$ and $\mathbb{B}^3 \setminus \phi(S_1)$ are linearly separable. Also, since $|S_2| = 2$ and the Hamming distance between two vectors belonging to S_2 is 1, S_2 satisfies the conditions in Lemma 3. Therefore $S_1 \setminus S_2 = S$ can be realized as the local pattern set.
   d) We assume without loss of generality that $d_{\mathrm{H}}(\phi(\boldsymbol{q}^{i_1}), \phi(\boldsymbol{q}^{i_3})) = 1$. Let $S_1 = S \cup \{\boldsymbol{q}^*\}$ where $\boldsymbol{q}^* \in \mathbb{B}^3$ satisfies $q_0^* = q_0^{i_3}$ and $d_{\mathrm{H}}(\phi(\boldsymbol{q}^{i_2}), \phi(\boldsymbol{q}^*)) = 1$ and let $S_2 = \{\boldsymbol{q}^*\}$. Then S_1 and S_2 satisfy the conditions in Lemma 3 and hence $S_1 \setminus S_2 = S$ can be realized as the local pattern set.
6) Let $S = \{\boldsymbol{q}^{i_1}, \boldsymbol{q}^{i_2}, \boldsymbol{q}^{i_3}, \boldsymbol{q}^{i_4}, \boldsymbol{q}^{i_5}\}$. If $\phi(S) \cup \{\boldsymbol{0}\}$ and $\mathbb{B}^3 \setminus \phi(S)$ are linearly separable, it follows from Theorem 1 that S can be realized by a CNN with $u_0 = 0$. We therefore assume hereafter that $\phi(S) \cup \{\boldsymbol{0}\}$ and $\mathbb{B}^3 \setminus \phi(S)$ are not linearly separable. We further assume without loss of generality that $q_0^{i_1} = q_0^{i_2} = q_0^{i_3} \neq q_0^{i_4} = q_0^{i_5}$. Let us first consider the case where $d_{\mathrm{H}}(\boldsymbol{q}^{i_4}, \boldsymbol{q}^{i_5}) = 1$. Let $S_1 = \{\boldsymbol{q} \in \mathbb{B}^3 \,|\, q_0 = q_0^{i_1}\} \cup \{\boldsymbol{q}^{i_4}, \boldsymbol{q}^{i_5}\}$ and $S_2 = S_1 \setminus S$. Then S_1 and S_2 satisfy the conditions in Lemma 3 and hence $S_1 \setminus S_2 = S$ can be realized as the local pattern set. Let us next consider the case where $d_{\mathrm{H}}(\boldsymbol{q}^{i_4}, \boldsymbol{q}^{i_5}) = 2$. This case is further divided into the subcases i) $d_{\mathrm{H}}(\boldsymbol{q}^{i_1}, \boldsymbol{q}^{i_2}) = d_{\mathrm{H}}(\boldsymbol{q}^{i_1}, \boldsymbol{q}^{i_3}) = d_{\mathrm{H}}(\boldsymbol{q}^{i_1}, \boldsymbol{q}^{i_4}) = 1$ and ii) $d_{\mathrm{H}}(\boldsymbol{q}^{i_1}, \boldsymbol{q}^{i_2}) = d_{\mathrm{H}}(\boldsymbol{q}^{i_1}, \boldsymbol{q}^{i_3}) = 1$ and $d_{\mathrm{H}}(\boldsymbol{q}^{i_1}, \boldsymbol{q}^{i_4}) = 2$. In Subcase i), let $S_1 = S \setminus \{\boldsymbol{q}^{i_5}\}$ and $S_2 = \{\boldsymbol{q}^{i_5}\}$. Since S_1 and S_2 satisfy the conditions in Lemma 4, $S_1 \cup S_2 = S$ can be realized by a CNN as the local pattern set. In Subcase ii), let $S_1 = S \cup \{\boldsymbol{q}^*\}$ and $S_2 = \{\boldsymbol{q}^*\}$ where $\boldsymbol{q}^*$ is the vector satisfying $\boldsymbol{q}^* \notin S$ and $d_{\mathrm{H}}(\boldsymbol{q}^{i_1}, \boldsymbol{q}^*) = 1$. Then S_1 and S_2 satisfy the conditions in Lemma 3 and hence $S_1 \setminus S_2 = S$ can be realized as the local pattern set.
7) Let S_1 be a subset of $\mathbb{B}^3$ such that $|S_1| = 7$ and $S \subset S_1$. Let $S_2 = S_1 \setminus S$. Then S_1 and S_2 satisfy the conditions in Lemma 3 and hence $S_1 \setminus S_2 = S$ can be realized as the local pattern set.

■

Lemma 5 shows that many but not all subsets of $\mathbb{B}^3$ can be realized as the local pattern set of a CNN described by (1). The cases not covered by Theorem 1 and Lemma 5 are summarized as follows:

1) $S = \{\boldsymbol{q}^{i_1}, \boldsymbol{q}^{i_2}\}$ where $q_0^{i_1} = q_0^{i_2}$ and $d_{\mathrm{H}}(\boldsymbol{q}^i, \boldsymbol{q}^j) = 2$.
2) $S = \{\boldsymbol{q}^{i_1}, \boldsymbol{q}^{i_2}, \boldsymbol{q}^{i_3}\}$ where $q_0^{i_1} = q_0^{i_2} \neq q_0^{i_3}$, $d_{\mathrm{H}}(\boldsymbol{q}^{i_1}, \boldsymbol{q}^{i_2}) = 2$ and $d_{\mathrm{H}}(\boldsymbol{q}^{i_1}, \boldsymbol{q}^{i_3}) = 1$.
3) $S = \{\boldsymbol{q}^{i_1}, \boldsymbol{q}^{i_2}, \boldsymbol{q}^{i_3}, \boldsymbol{q}^{i_4}\}$ where $q_0^{i_1} = q_0^{i_2} \neq q_0^{i_3} = q_0^{i_4}$ and $d_{\mathrm{H}}(\boldsymbol{q}^{i_1}, \boldsymbol{q}^{i_2}) = d_{\mathrm{H}}(\boldsymbol{q}^{i_3}, \boldsymbol{q}^{i_4}) = 2$.

Note that we do not have to consider the case where $S = \{\boldsymbol{q}^{i_1}, \boldsymbol{q}^{i_2}, \boldsymbol{q}^{i_3}\}$, $q_0^{i_1} = q_0^{i_2} \neq q_0^{i_3}$, $d_{\mathrm{H}}(\boldsymbol{q}^{i_1}, \boldsymbol{q}^{i_2}) = 2$ and $d_{\mathrm{H}}(\boldsymbol{q}^{i_1}, \boldsymbol{q}^{i_3}) = 2$ because these conditions lead to $d_{\mathrm{H}}(\boldsymbol{q}^{i_2}, \boldsymbol{q}^{i_3}) = 2$ which means this case falls into Case 4-c in Lemma 3. Also we do not have to consider the case where $S = \{\boldsymbol{q}^{i_1}, \boldsymbol{q}^{i_2}, \boldsymbol{q}^{i_3}\}$, $q_0^{i_1} = q_0^{i_2} \neq q_0^{i_3}$, $d_{\mathrm{H}}(\boldsymbol{q}^{i_1}, \boldsymbol{q}^{i_2}) = 2$ and $d_{\mathrm{H}}(\boldsymbol{q}^{i_1}, \boldsymbol{q}^{i_3}) = 3$ because these conditions lead to $d_{\mathrm{H}}(\boldsymbol{q}^{i_2}, \boldsymbol{q}^{i_3}) = 1$ which means this case falls into Case 2 above.

We will finally show that any S falls into one of these three classes can be realized as the local pattern set.

Lemma 6: Any $S = \{\boldsymbol{q}^{i_1}, \boldsymbol{q}^{i_2}\}$ such that $q_0^{i_1} = q_0^{i_2}$ and $d_{\mathrm{H}}(\boldsymbol{q}^{i_1}, \boldsymbol{q}^{i_2}) = 2$ can be realized as the local pattern set of a CNN described by (1).

Proof: We will prove this lemma only for $S = \{\boldsymbol{q}^2, \boldsymbol{q}^7\}$. Remaining three cases: $S = \{\boldsymbol{q}^0, \boldsymbol{q}^5\}$, $S = \{\boldsymbol{q}^1, \boldsymbol{q}^4\}$ and

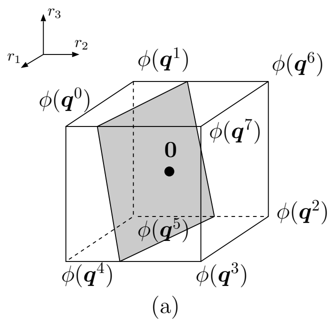

Fig. 7. How to realize $\{\boldsymbol{q}^2, \boldsymbol{q}^7\}$ as the local pattern set by using Lemma 6. (a) The plane defined by $\sigma([r_1, r_2, r_3]) = [0.2, 1, 0.2][r_1, r_2, r_3]^T + 1.2 = 1$. (b) Linear separation of S_2 and $\mathbb{B}^3 \setminus S_2$.

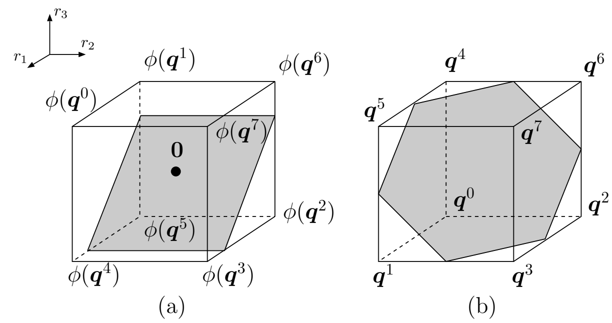

Fig. 8. How to realize $\{\boldsymbol{q}^0, \boldsymbol{q}^2, \boldsymbol{q}^5, \boldsymbol{q}^7\}$ as the local pattern set by using Lemma 8. (a) The plane defined by $\sigma([r_1, r_2, r_3]) = [0.2, 0, 0.2][r_1, r_2, r_3]^T + 1.1 = 1$. (b) Linear separation of S_2 and $\mathbb{B}^3 \setminus S_2$.

$S = \{\boldsymbol{q}^3, \boldsymbol{q}^6\}$ can be proved in a similar way. Let $S_2 = \{\boldsymbol{q}^3, \boldsymbol{q}^6, \boldsymbol{q}^7\}$. Since S_2 and $\mathbb{B}^3 \setminus S_2$ are linearly separable (see Fig.7(b)), it follows from Lemma 2 that there exist $\boldsymbol{d} = [d_{-1}, d_0, d_{+1}]$, $\hat{a}_0$ and $\hat{I}$ such that the differential equation (6) has a unique equilibrium point $\hat{x}^*$ for each $\boldsymbol{q}$ and $f(\hat{x}^*)$ is given by (11). Let $\boldsymbol{a} = [0.2, 1.2, 0.2]$, $u_0 = 0$ and $I = 1$. Then the values of the left-hand side of (5) for $\boldsymbol{q} = \boldsymbol{q}^i$, which is denoted as $\sigma(\phi(\boldsymbol{q}^i))$, for $i = 0, 1, \ldots, 7$ are $\sigma(\phi(\boldsymbol{q}^0)) = [a_{-1}, I, a_{+1}]\phi(\boldsymbol{q}^0)^T + a_0 + q_0 u_0 f(\hat{x}^*) = [0.2, 1, 0.2][1, -1, 1]^T + 1.2 + 0 = 0.6$, $\sigma(\phi(\boldsymbol{q}^1)) = 0.2$, $\sigma(\phi(\boldsymbol{q}^2)) = 1.8$, $\sigma(\phi(\boldsymbol{q}^3)) = 2.2$, $\sigma(\phi(\boldsymbol{q}^4)) = 0.2$, $\sigma(\phi(\boldsymbol{q}^5)) = -0.2$, $\sigma(\phi(\boldsymbol{q}^6)) = 2.2$ and $\sigma(\phi(\boldsymbol{q}^7)) = 2.6$ (see Fig.7(a) where the plane defined by $\sigma([r_1, r_2, r_3]) = 1$ is indicated by gray). Now we increase the values of u_0 and I by L where L is a constant whose value will be determined later. Then $\sigma(\phi(\boldsymbol{q}^i))$ increases by $2L$ for $i = 3, 6$ and 7, and remains the same for $i = 0, 1, 2, 4$ and 5. Therefore, by setting $L = -0.7$, we can make the value of $\sigma(\phi(\boldsymbol{q}^i))$ greater than 1 only for $i = 2$ and 7. This means that $S = \{\boldsymbol{q}^2, \boldsymbol{q}^7\}$ can be realized as the local pattern set. ■

Lemma 7: Any $S = \{\boldsymbol{q}^{i_1}, \boldsymbol{q}^{i_2}, \boldsymbol{q}^{i_3}\}$ such that $q_0^{i_1} = q_0^{i_2} \neq q_0^{i_3}$, $d_{\mathrm{H}}(\boldsymbol{q}^{i_1}, \boldsymbol{q}^{i_2}) = 2$ and $d_{\mathrm{H}}(\boldsymbol{q}^{i_1}, \boldsymbol{q}^{i_3}) = 1$ can be realized as the local pattern set of a CNN described by (1).

Proof: We will prove this lemma only for $S = \{\boldsymbol{q}^0, \boldsymbol{q}^2, \boldsymbol{q}^7\}$. Remaining seven cases: $S = \{\boldsymbol{q}^2, \boldsymbol{q}^5, \boldsymbol{q}^7\}$, $S = \{\boldsymbol{q}^1, \boldsymbol{q}^3, \boldsymbol{q}^6\}$, $S = \{\boldsymbol{q}^3, \boldsymbol{q}^4, \boldsymbol{q}^6\}$, $S = \{\boldsymbol{q}^0, \boldsymbol{q}^2, \boldsymbol{q}^5\}$, $S = \{\boldsymbol{q}^0, \boldsymbol{q}^5, \boldsymbol{q}^7\}$, $S = \{\boldsymbol{q}^1, \boldsymbol{q}^3, \boldsymbol{q}^4\}$ and $S = \{\boldsymbol{q}^1, \boldsymbol{q}^4, \boldsymbol{q}^6\}$ can be proved in a similar way. Let $S_2 = \{\boldsymbol{q}^3, \boldsymbol{q}^6, \boldsymbol{q}^7\}$. Since S_2 and $\mathbb{B}^3 \setminus S_2$ are linearly separable, it follows from Lemma 2 that there exist $\boldsymbol{d} = [d_{-1}, d_0, d_{+1}]$, $\hat{a}_0$ and $\hat{I}$ such that the differential equation (6) has a unique equilibrium point $\hat{x}^*$ for each $\boldsymbol{q}$ and $f(\hat{x}^*)$ is given by (11). Let $\boldsymbol{a} = [0.2, 1.8, 0.2]$, $u_0 = 0$ and $I = 1$. Then the values of the left-hand side of (5) for $\boldsymbol{q} = \boldsymbol{q}^i$, which is denoted as $\sigma(\phi(\boldsymbol{q}^i))$, for $i = 0, 1, \ldots, 7$ are $\sigma(\phi(\boldsymbol{q}^0)) = 1.2$, $\sigma(\phi(\boldsymbol{q}^1)) = 0.8$, $\sigma(\phi(\boldsymbol{q}^2)) = 2.4$, $\sigma(\phi(\boldsymbol{q}^3)) = 2.8$, $\sigma(\phi(\boldsymbol{q}^4)) = 0.8$, $\sigma(\phi(\boldsymbol{q}^5)) = 0.4$, $\sigma(\phi(\boldsymbol{q}^6)) = 2.8$ and $\sigma(\phi(\boldsymbol{q}^7)) = 3.2$. Now we increase the values of u_0 and I by L. Then $\sigma(\phi(\boldsymbol{q}^i))$ increases by $2L$ for $i = 3, 6$ and 7, and remains the same for $i = 0, 1, 2, 4$ and 5. Therefore, by setting $L = -1$, we can make the value of $\sigma(\phi(\boldsymbol{q}^i))$ greater than 1 only for $i = 0, 2$ and 7. This means that $S = \{\boldsymbol{q}^0, \boldsymbol{q}^2, \boldsymbol{q}^7\}$ can be realized as the local pattern set. ■

Lemma 8: Any $S = \{\boldsymbol{q}^{i_1}, \boldsymbol{q}^{i_2}, \boldsymbol{q}^{i_3}, \boldsymbol{q}^{i_4}\}$ such that $q_0^{i_1} = q_0^{i_2} \neq q_0^{i_3} = q_0^{i_4}$ and $d_{\mathrm{H}}(\boldsymbol{q}^{i_1}, \boldsymbol{q}^{i_2}) = d_{\mathrm{H}}(\boldsymbol{q}^{i_3}, \boldsymbol{q}^{i_4}) = 2$ can be realized as the local pattern set of a CNN described by (1).

Proof: There are four possible cases to be considered: i) $S = \{\boldsymbol{q}^0, \boldsymbol{q}^2, \boldsymbol{q}^5, \boldsymbol{q}^7\}$, ii) $S = \{\boldsymbol{q}^1, \boldsymbol{q}^3, \boldsymbol{q}^4, \boldsymbol{q}^6\}$, iii) $S = \{\boldsymbol{q}^0, \boldsymbol{q}^3, \boldsymbol{q}^5, \boldsymbol{q}^6\}$ and iv) $S = \{\boldsymbol{q}^1, \boldsymbol{q}^2, \boldsymbol{q}^4, \boldsymbol{q}^7\}$. We first consider Case i). Let $S_2 = \{\boldsymbol{q}^3, \boldsymbol{q}^5, \boldsymbol{q}^6, \boldsymbol{q}^7\}$. Since S_2 and $\mathbb{B}^3 \setminus S_2$ are linearly separable (see Fig.8(b)), it follows from Lemma 2 that there exist $\boldsymbol{d} = [d_{-1}, d_0, d_{+1}]$, $\hat{a}_0$ and $\hat{I}$ such that the differential equation (6) has a unique equilibrium point $\hat{x}^*$ for each $\boldsymbol{q}$ and $f(\hat{x}^*)$ is given by (11). Let $\boldsymbol{a} = [0.2, 1.1, 0.2]$ and $u_0 = I = 0$. Then the values of the left-hand side of (5) for $\boldsymbol{q} = \boldsymbol{q}^i$, which is denoted as $\sigma(\phi(\boldsymbol{q}^i))$, for $i = 0, 1, \ldots, 7$ are $\sigma(\phi(\boldsymbol{q}^0)) = 1.5$, $\sigma(\phi(\boldsymbol{q}^1)) = 1.1$, $\sigma(\phi(\boldsymbol{q}^2)) = 0.7$, $\sigma(\phi(\boldsymbol{q}^3)) = 1.1$, $\sigma(\phi(\boldsymbol{q}^4)) = 1.1$, $\sigma(\phi(\boldsymbol{q}^5)) = 0.7$, $\sigma(\phi(\boldsymbol{q}^6)) = 1.1$ and $\sigma(\phi(\boldsymbol{q}^7)) = 1.5$ (see Fig.8(a) where the plane defined by $\sigma([r_1, r_2, r_3]) = 1$ is indicated by gray). Now we increase the value of u_0 by L. Then $\sigma(\phi(\boldsymbol{q}^i))$ increases by L for $i = 0, 1, 4$ and 7, and decreases by L for $i = 2, 3, 5$ and 6. Therefore, by setting $L = -0.4$, we can make the value of $\sigma(\phi(\boldsymbol{q}^i))$ greater than 1 only for $i = 0, 2, 5$ and 7. This means that $S = \{\boldsymbol{q}^0, \boldsymbol{q}^2, \boldsymbol{q}^5, \boldsymbol{q}^7\}$ can be realized as the local pattern set. Next we consider Case iii). Let $S_2 = \{\boldsymbol{q}^5, \boldsymbol{q}^7\}$. Since S_2 and $\mathbb{B}^3 \setminus S_2$ are linearly separable, it follows from Lemma 1 that there exist $\boldsymbol{d} = [d_{-1}, d_0, d_{+1}]$, $\hat{a}_0$ and $\hat{I}$ such that the differential equation (6) has a unique equilibrium point $\hat{x}^*$ for each $\boldsymbol{q}$ and $f(\hat{x}^*)$ is given by (11). Let $\boldsymbol{a} = [0.2, 1.1, 0.2]$, $u_0 = 0$ and $I = -0.2$. Then we have $\sigma(\phi(\boldsymbol{q}^0)) = 1.7$, $\sigma(\phi(\boldsymbol{q}^1)) = 1.3$, $\sigma(\phi(\boldsymbol{q}^2)) = 0.5$, $\sigma(\phi(\boldsymbol{q}^3)) = 0.9$, $\sigma(\phi(\boldsymbol{q}^4)) = 1.3$, $\sigma(\phi(\boldsymbol{q}^5)) = 0.9$, $\sigma(\phi(\boldsymbol{q}^6)) = 0.9$ and $\sigma(\phi(\boldsymbol{q}^7)) = 1.3$. Now we increase the value of u_0 by L. Then $\sigma(\phi(\boldsymbol{q}^i))$ increases by L for $i = 0, 1, 4$ and 7, and decreases by L for $i = 2, 3, 5$ and 6. Therefore, by setting $L = -0.4$, we can make the value of $\sigma(\phi(\boldsymbol{q}^i))$ greater than 1 only for $i = 0, 3, 5$ and 6. This means that $S = \{\boldsymbol{q}^0, \boldsymbol{q}^3, \boldsymbol{q}^5, \boldsymbol{q}^6\}$ can be realized as the local pattern set. Cases ii) and iv) can be proved in a similar way as Cases i) and iii) respectively. ■

C. Summary and Comparison to Related Work

Let us first summarize the approach taken in the proof of Theorem 2. For given $S \subseteq \mathbb{B}^3$, we first separate $\{\phi(\boldsymbol{q}^i)\}_{i=0}^7$ into two classes by the plane $[a_{-1}, I, a_{+1}][r_1, r_2, r_3]^T + a_0 - 1 = 0$, and $\{\boldsymbol{q}^i\}_{i=0}^7$ into two classes by the plane $\boldsymbol{d}[r_1, r_2, r_3]^T + \hat{I} = 0$. We then adjust the parameter values so that

$$\mathrm{sgn}\left([a_{-1}, I, a_{+1}]\phi(\boldsymbol{q})^T + a_0 - 1 + q_0 u_0 \mathrm{sgn}\left(\boldsymbol{d}\boldsymbol{q}^T + \hat{I}\right)\right) \tag{14}$$

takes 1 if and only if $\boldsymbol{q} \in S$, where $\mathrm{sgn}(u)$ takes 1 if $u > 0$, 0 if $u = 0$, and -1 if $u < 0$. The role of the second layer is represented by the term $q_0 u_0 \mathrm{sgn}(\boldsymbol{d}\boldsymbol{q}^T + \hat{I})$. Owing to this term, the separation (14) becomes nonlinear and we can realize any subset of $\mathbb{B}^3$ as the local pattern set.

The most important step in our approach is how to separate $\{\boldsymbol{q}^i\}_{i=0}^7$ into two classes S_2 and $\{\boldsymbol{q}^i\}_{i=0}^7 \setminus S_2$ by the plane $\boldsymbol{d}[r_1, r_2, r_3]^T + \hat{I} = 0$. We have determined this linear separation manually for each case, but this is possible only when the number of cases to be considered is small. If we apply our approach to more general case, it will be necessary to develop a systematic way to find the linear separation.

Let us next compare the results of this paper to related work. Theorem 1 is related to some results on equilibrium analysis of single-layer CNNs [12], [17]–[20]. These results are general in the sense that they can be applied to general two-dimensional CNNs. On the other hand, only a class of simple one-dimensional CNNs is considered in Theorem 1. However, this allows us to provide a complete characterization of the set of stable patterns that can be realized by those CNNs.

Nonlinear separation of binary vectors via CNNs was also studied by Dogaru and Chua [10]. They considered the problem of separating n-dimensional binary vectors with a class of piecewise-linear functions called multi-nested discriminant functions, and showed that all of 2^{2^n} separation can be realized for $n \leq 4$. However, they did not prove it analytically but just presented parameter values which were found by a computer program. In this paper, on the other hand, we have given an analytical proof of Theorem 2 from which we can see how each subset of $\mathbb{B}^3$ is realized as the local pattern set.

IV. Template Optimization

We have shown in the previous section that for every subset S of $\mathbb{B}^3$ there exists a set of templates such that S is the local pattern set of the CNN. However, the optimality of the templates was not considered at all. In this section, we propose a simple method to find the template values which maximize the robustness, and present the obtained values for all of 256 subsets of $\mathbb{B}^3$.

Let us first consider 59 subsets of $\mathbb{B}^3$ which can be realized as the local pattern set of a CNN with $u_0 = 0$. In these cases, we only have to determine the values of $\boldsymbol{a} = [a_{-1}, a_0, a_{+1}]$ and I. By normalizing the value of $a_0 - 1$ to 1, we can express the conditions for a subset S of $\mathbb{B}^3$ to be the local pattern set of a CNN with $u_0 = 0$ as follows:

$$[a_{-1}, I, a_{+1}]\, \phi(\boldsymbol{q})^T + 1 \begin{cases} > 0, & \forall \boldsymbol{q} \in S \\ < 0, & \forall \boldsymbol{q} \in \mathbb{B}^3 \setminus S \end{cases} \tag{15}$$

When a CNN is implemented with the analog circuit, the template values cannot be realized exactly but suffer from perturbations [21], [22]. It is thus important to make the set of templates robust against the perturbation [21]–[26]. A simple approach to finding robust templates is to maximize

$$\min_{\boldsymbol{q} \in \mathbb{B}^3} |[a_{-1}, I, a_{+1}]\, \phi(\boldsymbol{q})^T + 1|$$

within a certain parameter region, which is formulated as the following linear programming problem.

Problem 1: Maximize δ_1 subject to

$$\begin{gathered} [a_{-1}, I, a_{+1}]\, \phi(\boldsymbol{q})^T + 1 \geq \delta_1,\ \forall \boldsymbol{q} \in S \\ [a_{-1}, I, a_{+1}]\, \phi(\boldsymbol{q})^T + 1 \leq -\delta_1,\ \forall \boldsymbol{q} \in \mathbb{B}^3 \setminus S \\ |a_{-1}| \leq U_1,\ |a_{+1}| \leq U_1,\ |I| \leq U_1,\ \delta_1 \geq 0 \end{gathered}$$

where U_1 is a positive constant.

Note that Problem 1 does not always have a feasible solution because the second constraint will not be satisfied if U_1 is too small. We have verified with numerical calculations that Problem 1 has a feasible solution if $U_1 \geq 1$.

Let us next consider the remaining 197 subsets of $\mathbb{B}^3$ which cannot be realized as the local pattern set of a CNN with $u_0 = 0$. In each of these 197 cases, we first choose the set $S_2 \subseteq \mathbb{B}^3$ such that S_2 and $\mathbb{B}^3 \setminus S_2$ are linearly separable, as shown in the proofs of Lemmas 4-8. We then determine the values of the parameters $\boldsymbol{a} = [a_{-1}, a_0, a_{+1}]$, u_0 and I so that

$$[a_{-1}, I, a_{+1}]\, \phi(\boldsymbol{q})^T + a_0 + q_0 u_0 \hat{y}^*(\boldsymbol{q}) \begin{cases} > 1, & \forall \boldsymbol{q} \in S \\ < 1, & \forall \boldsymbol{q} \in \mathbb{B}^3 \setminus S \end{cases} \tag{16}$$

is satisfied where $\hat{y}^*(\boldsymbol{q})$ is defined as follows:

$$\hat{y}^*(\boldsymbol{q}) = \begin{cases} 1, & \text{if } \boldsymbol{q} \in S_2 \\ -1, & \text{if } \boldsymbol{q} \in \mathbb{B}^3 \setminus S_2 \end{cases} \tag{17}$$

We also determine the values of the parameters $\hat{a}_0$, $\boldsymbol{d} = [d_{-1}, d_0, d_{+1}]$ and $\hat{I}$ so that the differential equation (6) has a unique equilibrium point $\hat{x}^*$ satisfying (11). According to Lemma 2, the value of $\hat{a}_0$ is set to $1+\epsilon$ where ϵ is a sufficiently small positive number, and the values of the parameters $\boldsymbol{d}$ and $\hat{I}$ are determined so that

$$\boldsymbol{d}\, \boldsymbol{q}^T + \hat{I} \begin{cases} > 0, & \forall \boldsymbol{q} \in S_2 \\ < 0, & \forall \boldsymbol{q} \in \mathbb{B}^3 \setminus S_2 \end{cases} \tag{18}$$

is satisfied. Taking the robustness of the parameters against perturbation into account, we derive the following linear programming problems from (16) and (18).

Problem 2: Maximize δ_1 subject to

$$\begin{gathered} [a_{-1}, I, a_{+1}]\, \phi(\boldsymbol{q})^T + 1 + q_0 u_0 \hat{y}^*(\boldsymbol{q}) \geq \delta_1,\ \forall \boldsymbol{q} \in S \\ [a_{-1}, I, a_{+1}]\, \phi(\boldsymbol{q})^T + 1 + q_0 u_0 \hat{y}^*(\boldsymbol{q}) \leq -\delta_1,\ \forall \boldsymbol{q} \in \mathbb{B}^3 \setminus S \\ |a_{-1}| \leq U_1,\ |a_{+1}| \leq U_1,\ |I| \leq U_1,\ |u_0| \leq U_1,\ \delta_1 \geq 0 \end{gathered}$$

where U_1 is a positive constant.

Problem 3: Maximize δ_2 subject to

$$\begin{gathered} \boldsymbol{d}\, \boldsymbol{q}^T + \hat{I} \geq \delta_2,\ \forall \boldsymbol{q} \in S_2 \\ \boldsymbol{d}\, \boldsymbol{q}^T + \hat{I} \leq -\delta_2,\ \forall \boldsymbol{q} \in \mathbb{B}^3 \setminus S_2 \\ |d_{-1}| \leq U_2,\ |d_0| \leq U_2,\ |d_{+1}| \leq U_2,\ |\hat{I}| \leq U_2,\ \delta_2 \geq 0 \end{gathered}$$

where U_2 is a positive constant.

Note that in deriving Problem 2 the value of $a_0 - 1$ was set to 1 for the purpose of normalization. Note also that Problem 2 does not always have a feasible solution because the second constraint will not be satisfied if U_1 is too small. We have verified with numerical calculations that Problem 2 has a feasible solution if $U_1 \geq 2$.

Tables I-VI show the optimal values of the templates for all of 256 subsets of $\mathbb{B}^3$, which are derived by solving Problem 1 with $U_1 = 3$ and Problems 2 and 3 with $U_1 = 3$ and $U_2 = 1$. The first column is the ID of each subset S of $\mathbb{B}^3$ which is expressed by an integer between 0 and 255. More specifically, ID number is determined by $\text{ID} = \sum_{i=0}^{7} 2^i b_i$ where $b_i = 1$ if $\boldsymbol{q}^i \in S$ and $b_i = 0$ if $\boldsymbol{q}^i \in \mathbb{B}^3 \setminus S$. The second column is $S \subseteq \mathbb{B}^3$ itself; The value 1 (0, resp.) under an integer $i \in \{0, 1, \ldots, 7\}$ indicates that $\boldsymbol{q}^i$ belongs (does not belong, resp.) to S. For example, 01100100 means $S = \{\boldsymbol{q}^1, \boldsymbol{q}^2, \boldsymbol{q}^5\}$. The third column is $S_2 \subseteq \mathbb{B}^3$ which appears in (17) and Problem 3. The meanings of 1 and 0 are same as the second column. A blank entry in S_2 column means the set S can be realized by a single-layer CNN. There are 59 blank entries as we have seen in Theorem 1. The 4th to 13th columns are the optimal values of the templates. The 14th and 15th columns are the optimal values of δ_1 in Problems 1 and 2, and δ_2 in Problem 3, respectively. As shown in the table, the minimum values of δ_1 in Problem 1, δ_1 in Problem 2 and δ_2 in Problem 3 are 1, 0.5 and 0.5, respectively, which means that high level robustness was achieved by the proposed method.

Finally we should note that in general S_2 is not uniquely determined for each of 197 subsets of $\mathbb{B}^3$. Therefore the template values in Tables I-VI are optimal as far as S_2 is chosen as in the third column. In other words, it is possible that a larger value of δ_1 is obtained if a different S_2 is employed.

V. Conclusion

As an attempt to make clear the signal processing capability of two-layer CNNs, the variety of local pattern sets realized by a class of one-dimensional two-layer CNNs was studied. It was shown that any of 256 subsets of $\mathbb{B}^3$ can be realized as the local pattern set of such a CNN, while only 59 can be realized by one-dimensional single-layer CNNs. Also, a simple way to optimize the template values was proposed, which is formulated as a set of linear programming problems, and the obtained values were presented for all of 256 sets.

Since dynamical behavior of CNNs is not considered in this paper, the results cannot be directly applied to signal processing. However, Theorem 2 indicates the potential of two-layer CNNs in, for example, associative memories, because two-layer CNNs can store much more patterns than one-layer CNNs. Also, since it follows from Theorem 2 that the variety of stable patterns of one-dimensional two-layer CNNs is as diverse as one-dimensional cellular automata [27], it may be possible to apply two-layer CNNs to the modeling of various nonlinear phenomena.

Analysis of the dynamical behavior of the designed one-dimensional CNNs and the generalization of the results to two-dimensional case are future problems.

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TABLE I

OPTIMAL VALUES OF THE TEMPLATES (PART 1 OF 4).

ID	S	S_2												
	01234567	01234567	a_{-1}	a_0	a_{+1}	u_0	I	$\hat{a}_0$	d_{-1}	d_0	d_1	$\hat{I}$	δ_1	δ_2
0	00000000	00110011	0	2	0	−3	0	$1+\epsilon$	0	1	0	0	2	1
1	10000000	01001100	0	2	0	3	0	$1+\epsilon$	0.5	−1	0.5	−0.5	2	0.5
2	01000000	10001100	0	2	0	3	0	$1+\epsilon$	−0.5	−1	0.5	−0.5	2	0.5
3	11000000	00001100	0	2	0	3	0	$1+\epsilon$	0	−1	1	−1	2	1
4	00100000	00010011	0	2	0	−3	0	$1+\epsilon$	0.5	1	0.5	−0.5	2	0.5
5	10100000	00010011	1	2	1	−3	−1	$1+\epsilon$	0.5	1	0.5	−0.5	1	0.5
6	01100000	00010011	−1	2	1	−3	−1	$1+\epsilon$	0.5	1	0.5	−0.5	1	0.5
7	11100000	00001100	−1	2	−1	3	1	$1+\epsilon$	0	−1	1	−1	1	1
8	00010000	00100011	0	2	0	−3	0	$1+\epsilon$	−0.5	1	0.5	−0.5	2	0.5
9	10010000	00100011	1	2	1	−3	−1	$1+\epsilon$	−0.5	1	0.5	−0.5	1	0.5
10	01010000	00100011	−1	2	1	−3	−1	$1+\epsilon$	−0.5	1	0.5	−0.5	1	0.5
11	11010000	00001100	1	2	−1	3	1	$1+\epsilon$	0	−1	1	−1	1	1
12	00110000	00000011	0	2	0	−3	0	$1+\epsilon$	0	1	1	−1	2	1
13	10110000	00000011	1	2	1	−3	−1	$1+\epsilon$	0	1	1	−1	1	1
14	01110000	00000011	−1	2	1	−3	−1	$1+\epsilon$	0	1	1	−1	1	1
15	11110000	00001100	0	2	−1	2	1	$1+\epsilon$	0	−1	1	−1	1	1
16	00001000	11000100	0	2	0	3	0	$1+\epsilon$	0.5	−1	−0.5	−0.5	2	0.5
17	10001000	01000100	0	2	0	3	0	$1+\epsilon$	1	−1	0	−1	2	1
18	01001000	10001100	1	2	−1	3	−1	$1+\epsilon$	−0.5	−1	0.5	−0.5	1	0.5
19	11001000	00000100	0	2	0	3	0	$1+\epsilon$	0.5	−0.5	0.5	−1	2	0.5
20	00101000	00010011	1	2	−1	−3	−1	$1+\epsilon$	0.5	1	0.5	−0.5	1	0.5
21	10101000	01000100	−1	2	−1	3	1	$1+\epsilon$	1	−1	0	−1	1	1
22	01101000	00000100	−1.5	2	−1.5	3	1.5	$1+\epsilon$	0.5	−0.5	0.5	−1	0.5	0.5
23	11101000	00000100	−1	2	−1	3	1	$1+\epsilon$	0.5	−0.5	0.5	−1	1	0.5
24	00011000	00100011	2	2	−2	−3	0	$1+\epsilon$	−0.5	1	0.5	−0.5	2	0.5
25	10011000	01000100	2	2	−2	3	0	$1+\epsilon$	1	−1	0	−1	2	1
26	01011000	10001100	1	2	−1	2	0	$1+\epsilon$	−0.5	−1	0.5	−0.5	1	0.5
27	11011000	00000100	1	2	−1	2	0	$1+\epsilon$	0.5	−0.5	0.5	−1	1	0.5
28	00111000	00000011	2	2	−2	−3	0	$1+\epsilon$	0	1	1	−1	2	1
29	10111000	01000100	0	2	−1	2	1	$1+\epsilon$	1	−1	0	−1	1	1
30	01111000	00000100	−1	2	−2	3	2	$1+\epsilon$	0.5	−0.5	0.5	−1	1	0.5
31	11111000	00000100	0	2	−1	2	1	$1+\epsilon$	0.5	−0.5	0.5	−1	1	0.5
32	00000100	11001000	0	2	0	3	0	$1+\epsilon$	−0.5	−1	−0.5	−0.5	2	0.5
33	10000100	01001100	−1	2	−1	3	−1	$1+\epsilon$	0.5	−1	0.5	−0.5	1	0.5
34	01000100	10001000	0	2	0	3	0	$1+\epsilon$	−1	−1	0	−1	2	1
35	11000100	00001000	0	2	0	3	0	$1+\epsilon$	−0.5	−0.5	0.5	−1	2	0.5
36	00100100	00010011	−2	2	−2	−3	0	$1+\epsilon$	0.5	1	0.5	−0.5	2	0.5
37	10100100	01001100	−1	2	−1	2	0	$1+\epsilon$	0.5	−1	0.5	−0.5	1	0.5
38	01100100	10001000	−2	2	−2	3	0	$1+\epsilon$	−1	−1	0	−1	2	1
39	11100100	00001000	−1	2	−1	2	0	$1+\epsilon$	−0.5	−0.5	0.5	−1	1	0.5
40	00010100	00100011	−1	2	−1	−3	−1	$1+\epsilon$	−0.5	1	0.5	−0.5	1	0.5
41	10010100	00001000	1.5	2	−1.5	3	1.5	$1+\epsilon$	−0.5	−0.5	0.5	−1	0.5	0.5
42	01010100	10001000	1	2	−1	3	1	$1+\epsilon$	−1	−1	0	−1	1	1
43	11010100	00001000	1	2	−1	3	1	$1+\epsilon$	−0.5	−0.5	0.5	−1	1	0.5
44	00110100	00000011	−2	2	−2	−3	0	$1+\epsilon$	0	1	1	−1	2	1
45	10110100	00001000	1	2	−2	3	2	$1+\epsilon$	−0.5	−0.5	0.5	−1	1	0.5
46	01110100	10001000	0	2	−1	2	1	$1+\epsilon$	−1	−1	0	−1	1	1
47	11110100	00001000	0	2	−1	2	1	$1+\epsilon$	−0.5	−0.5	0.5	−1	1	0.5
48	00001100	11000000	0	2	0	3	0	$1+\epsilon$	0	−1	−1	−1	2	1
49	10001100	01000000	0	2	0	3	0	$1+\epsilon$	0.5	−0.5	−0.5	−1	2	0.5
50	01001100	10000000	0	2	0	3	0	$1+\epsilon$	−0.5	−0.5	−0.5	−1	2	0.5
51	11001100		0	2	0	0	−3						2	
52	00101100	11000000	−2	2	−2	3	0	$1+\epsilon$	0	−1	−1	−1	2	1
53	10101100	01000000	−1	2	−1	2	0	$1+\epsilon$	0.5	−0.5	−0.5	−1	1	0.5
54	01101100		−3	2	−3	0	−3						2	
55	11101100		−1	2	−1	0	−2						1	
56	00011100	11000000	2	2	−2	3	0	$1+\epsilon$	0	−1	−1	−1	2	1
57	10011100		3	2	−3	0	−3						2	
58	01011100	10000000	1	2	−1	2	0	$1+\epsilon$	−0.5	−0.5	−0.5	−1	1	0.5
59	11011100		1	2	−1	0	−2						1	
60	00111100		0	2	−3	0	0						2	
61	10111100		1	2	−2	0	−1						1	
62	01111100		−1	2	−2	0	−1						1	
63	11111100		0	2	−1	0	−1						1	

TABLE II

OPTIMAL VALUES OF THE TEMPLATES (PART 2 OF 4).

ID	S 0 1 2 3 4 5 6 7	S_2 0 1 2 3 4 5 6 7	a_{-1}	a_0	a_{+1}	u_0	I	$\hat{a}_0$	d_{-1}	d_0	d_1	$\hat{I}$	δ_1	δ_2
64	00000010	00110001	0	2	0	-3	0	$1+\epsilon$	0.5	1	-0.5	-0.5	2	0.5
65	10000010	00110001	1	2	1	-3	-1	$1+\epsilon$	0.5	1	-0.5	-0.5	1	0.5
66	01000010	00110001	-2	2	2	-3	0	$1+\epsilon$	0.5	1	-0.5	-0.5	2	0.5
67	11000010	00001100	-2	2	2	3	0	$1+\epsilon$	0	-1	1	-1	2	1
68	00100010	00010001	0	2	0	-3	0	$1+\epsilon$	1	1	0	-1	2	1
69	10100010	00010001	1	2	1	-3	-1	$1+\epsilon$	1	1	0	-1	1	1
70	01100010	00010001	-2	2	2	-3	0	$1+\epsilon$	1	1	0	-1	2	1
71	11100010	00001100	-1	2	0	2	1	$1+\epsilon$	0	-1	1	-1	1	1
72	00010010	00100011	-1	2	1	-3	1	$1+\epsilon$	-0.5	1	0.5	-0.5	1	0.5
73	10010010	00000001	1.5	2	1.5	-3	-1.5	$1+\epsilon$	0.5	0.5	0.5	-1	0.5	0.5
74	01010010	00100011	-1	2	1	-2	0	$1+\epsilon$	-0.5	1	0.5	-0.5	1	0.5
75	11010010	00000001	1	2	2	-3	-2	$1+\epsilon$	0.5	0.5	0.5	-1	1	0.5
76	00110010	00000001	0	2	0	-3	0	$1+\epsilon$	0.5	0.5	0.5	-1	2	0.5
77	10110010	00000001	1	2	1	-3	-1	$1+\epsilon$	0.5	0.5	0.5	-1	1	0.5
78	01110010	00000001	-1	2	1	-2	0	$1+\epsilon$	0.5	0.5	0.5	-1	1	0.5
79	11110010	00000001	0	2	1	-2	-1	$1+\epsilon$	0.5	0.5	0.5	-1	1	0.5
80	00001010	00110001	1	2	-1	-3	-1	$1+\epsilon$	0.5	1	-0.5	-0.5	1	0.5
81	10001010	01000100	-1	2	1	3	1	$1+\epsilon$	1	-1	0	-1	1	1
82	01001010	11000100	-1	2	1	2	0	$1+\epsilon$	0.5	-1	-0.5	-0.5	1	0.5
83	11001010	00000100	-1	2	1	2	0	$1+\epsilon$	0.5	-0.5	0.5	-1	1	0.5
84	00101010	00010001	1	2	-1	-3	-1	$1+\epsilon$	1	1	0	-1	1	1
85	10101010	01000100	-1	2	0	2	1	$1+\epsilon$	1	-1	0	-1	1	1
86	01101010	00000100	-2	2	-1	3	2	$1+\epsilon$	0.5	-0.5	0.5	-1	1	0.5
87	11101010	00000100	-1	2	0	2	1	$1+\epsilon$	0.5	-0.5	0.5	-1	1	0.5
88	00011010	00110001	1	2	-1	-2	0	$1+\epsilon$	0.5	1	-0.5	-0.5	1	0.5
89	10011010	00000001	2	2	1	-3	-2	$1+\epsilon$	0.5	0.5	0.5	-1	1	0.5
90	01011010	01110001	1	2	-1	-2	0	$1+\epsilon$	1	1	-1	0	1	1
91	11011010	00000001	1	2	1	-2	-2	$1+\epsilon$	0.5	0.5	0.5	-1	1	0.5
92	00111010	00000001	1	2	-1	-2	0	$1+\epsilon$	0.5	0.5	0.5	-1	1	0.5
93	10111010	00000001	1	2	0	-2	-1	$1+\epsilon$	0.5	0.5	0.5	-1	1	0.5
94	01111010	00000100	-1	2	-1	2	2	$1+\epsilon$	0.5	-0.5	0.5	-1	1	0.5
95	11111010	00000100	-0.5	2	-0.5	1.5	1	$1+\epsilon$	0.5	-0.5	0.5	-1	0.5	0.5
96	00000110	00110001	-1	2	-1	-3	-1	$1+\epsilon$	0.5	1	-0.5	-0.5	1	0.5
97	10000110	01000000	-1.5	2	1.5	3	1.5	$1+\epsilon$	0.5	-0.5	-0.5	-1	0.5	0.5
98	01000110	10001000	-2	2	2	3	0	$1+\epsilon$	-1	-1	0	-1	2	1
99	11000110		-3	2	3	0	-3						2	
100	00100110	00010001	-2	2	-2	-3	0	$1+\epsilon$	1	1	0	-1	2	1
101	10100110	01000000	-2	2	1	3	2	$1+\epsilon$	0.5	-0.5	-0.5	-1	1	0.5
102	01100110		-3	2	0	0	0						2	
103	11100110		-2	2	1	0	-1						1	
104	00010110	00100000	-1.5	2	-1.5	-3	-1.5	$1+\epsilon$	-0.5	0.5	-0.5	-1	0.5	0.5
105	10010110	00000101	1.5	2	1.5	-3	-1.5	$1+\epsilon$	1	0	1	-1	0.5	1
106	01010110	00100000	-2	2	-1	-3	-2	$1+\epsilon$	-0.5	0.5	-0.5	-1	1	0.5
107	11010110	00010000	-2	2	2	3	1	$1+\epsilon$	0.5	0.5	-0.5	-1	1	0.5
108	00110110		-3	2	-3	0	3						2	
109	10110110	10000000	-2	2	-2	-3	-1	$1+\epsilon$	-0.5	-0.5	-0.5	-1	1	0.5
110	01110110		-2	2	-1	0	1						1	
111	11110110	00001000	-0.5	2	-0.5	3	2.5	$1+\epsilon$	-0.5	-0.5	0.5	-1	0.5	0.5
112	00001110	11000000	-1	2	1	3	1	$1+\epsilon$	0	-1	-1	-1	1	1
113	10001110	01000000	-1	2	1	3	1	$1+\epsilon$	0.5	-0.5	-0.5	-1	1	0.5
114	01001110	10000000	-1	2	1	2	0	$1+\epsilon$	-0.5	-0.5	-0.5	-1	1	0.5
115	11001110		-1	2	1	0	-2						1	
116	00101110	00010001	0	2	-1	-2	-1	$1+\epsilon$	1	1	0	-1	1	1
117	10101110	01000000	-1	2	0	2	1	$1+\epsilon$	0.5	-0.5	-0.5	-1	1	0.5
118	01101110		-2	2	-1	0	-1						1	
119	11101110		-1	2	0	0	-1						1	
120	00011110	00100000	-1	2	-2	-3	-2	$1+\epsilon$	-0.5	0.5	-0.5	-1	1	0.5
121	10011110	00000010	2	2	-2	3	1	$1+\epsilon$	-0.5	0.5	0.5	-1	1	0.5
122	01011110	00100000	-1	2	-1	-2	-2	$1+\epsilon$	-0.5	0.5	-0.5	-1	1	0.5
123	11011110	00100000	-0.5	2	-0.5	-1	-1.5	$1+\epsilon$	-0.5	0.5	-0.5	-1	0.5	0.5
124	00111110		-1	2	-2	0	1						1	
125	10111110	01000000	-0.5	2	-0.5	1	0.5	$1+\epsilon$	0.5	-0.5	-0.5	-1	0.5	0.5
126	01111110		-1	2	-1	0	0						1	
127	11111110		-0.5	2	-0.5	0	-0.5						0.5	

TABLE III

Optimal values of the templates (part 3 of 4).

ID	S	S_2												
	01234567	01234567	a_{-1}	a_0	a_{+1}	u_0	I	$\hat{a}_0$	d_{-1}	d_0	d_1	$\hat{I}$	δ_1	δ_2
128	00000001	00110010	0	2	0	−3	0	$1+\epsilon$	−0.5	1	−0.5	−0.5	2	0.5
129	10000001	00110010	2	2	2	−3	0	$1+\epsilon$	−0.5	1	−0.5	−0.5	2	0.5
130	01000001	00110010	−1	2	1	−3	−1	$1+\epsilon$	−0.5	1	−0.5	−0.5	1	0.5
131	11000001	00001100	2	2	2	3	0	$1+\epsilon$	0	−1	1	−1	2	1
132	00100001	00010011	1	2	1	−3	1	$1+\epsilon$	0.5	1	0.5	−0.5	1	0.5
133	10100001	00010011	1	2	1	−2	0	$1+\epsilon$	0.5	1	0.5	−0.5	1	0.5
134	01100001	00000010	−1.5	2	1.5	−3	−1.5	$1+\epsilon$	−0.5	0.5	0.5	−1	0.5	0.5
135	11100001	00000010	−1	2	2	−3	−2	$1+\epsilon$	−0.5	0.5	0.5	−1	1	0.5
136	00010001	00100010	0	2	0	−3	0	$1+\epsilon$	−1	1	0	−1	2	1
137	10010001	00100010	2	2	2	−3	0	$1+\epsilon$	−1	1	0	−1	2	1
138	01010001	00100010	−1	2	1	−3	−1	$1+\epsilon$	−1	1	0	−1	1	1
139	11010001	00001100	1	2	0	2	1	$1+\epsilon$	0	−1	1	−1	1	1
140	00110001	00000010	0	2	0	−3	0	$1+\epsilon$	−0.5	0.5	0.5	−1	2	0.5
141	10110001	00000010	1	2	1	−2	0	$1+\epsilon$	−0.5	0.5	0.5	−1	1	0.5
142	01110001	00000010	−1	2	1	−3	−1	$1+\epsilon$	−0.5	0.5	0.5	−1	1	0.5
143	11110001	00000010	0	2	1	−2	−1	$1+\epsilon$	−0.5	0.5	0.5	−1	1	0.5
144	00001001	00110010	1	2	−1	−3	−1	$1+\epsilon$	−0.5	1	−0.5	−0.5	1	0.5
145	10001001	01000100	2	2	2	3	0	$1+\epsilon$	1	−1	0	−1	2	1
146	01001001	10000000	1.5	2	1.5	3	1.5	$1+\epsilon$	−0.5	−0.5	−0.5	−1	0.5	0.5
147	11001001		3	2	3	0	−3						2	
148	00101001	00010000	1.5	2	−1.5	−3	−1.5	$1+\epsilon$	0.5	0.5	−0.5	−1	0.5	0.5
149	10101001	00010000	2	2	−1	−3	−2	$1+\epsilon$	0.5	0.5	−0.5	−1	1	0.5
150	01101001	10100000	1.5	2	1.5	3	1.5	$1+\epsilon$	−1	0	−1	−1	0.5	1
151	11101001	00100000	2	2	2	3	1	$1+\epsilon$	−0.5	0.5	−0.5	−1	1	0.5
152	00011001	00100010	2	2	−2	−3	0	$1+\epsilon$	−1	1	0	−1	2	1
153	10011001		3	2	0	0	0						2	
154	01011001	10000000	2	2	1	3	2	$1+\epsilon$	−0.5	−0.5	−0.5	−1	1	0.5
155	11011001		2	2	1	0	−1						1	
156	00111001		3	2	−3	0	3						2	
157	10111001		2	2	−1	0	1						1	
158	01111001	01000000	2	2	−2	−3	−1	$1+\epsilon$	0.5	−0.5	−0.5	−1	1	0.5
159	11111001	00000100	0.5	2	−0.5	1	0.5	$1+\epsilon$	0.5	−0.5	0.5	−1	0.5	0.5
160	00000101	00110010	−1	2	−1	−3	−1	$1+\epsilon$	−0.5	1	−0.5	−0.5	1	0.5
161	10000101	11001000	1	2	1	2	0	$1+\epsilon$	−0.5	−1	−0.5	−0.5	1	0.5
162	01000101	10001000	1	2	1	3	1	$1+\epsilon$	−1	−1	0	−1	1	1
163	11000101	00001000	1	2	1	2	0	$1+\epsilon$	−0.5	−0.5	0.5	−1	1	0.5
164	00100101	00110010	−1	2	−1	−2	0	$1+\epsilon$	−0.5	1	−0.5	−0.5	1	0.5
165	10100101	00010111	1	2	1	−2	0	$1+\epsilon$	1	1	1	0	1	1
166	01100101	00000010	−2	2	1	−3	−2	$1+\epsilon$	−0.5	0.5	0.5	−1	1	0.5
167	11100101	00000010	−1	2	1	−2	−2	$1+\epsilon$	−0.5	0.5	0.5	−1	1	0.5
168	00010101	00100010	−1	2	−1	−3	−1	$1+\epsilon$	−1	1	0	−1	1	1
169	10010101	00001000	2	2	−1	3	2	$1+\epsilon$	−0.5	−0.5	0.5	−1	1	0.5
170	01010101	10001000	1	2	0	2	1	$1+\epsilon$	−1	−1	0	−1	1	1
171	11010101	00001000	1	2	0	2	1	$1+\epsilon$	−0.5	−0.5	0.5	−1	1	0.5
172	00110101	00000010	−1	2	−1	−2	0	$1+\epsilon$	−0.5	0.5	0.5	−1	1	0.5
173	10110101	00001000	1	2	−1	2	2	$1+\epsilon$	−0.5	−0.5	0.5	−1	1	0.5
174	01110101	00000010	−1	2	0	−2	−1	$1+\epsilon$	−0.5	0.5	0.5	−1	1	0.5
175	11110101	00001000	0.5	2	−0.5	1.5	1	$1+\epsilon$	−0.5	−0.5	0.5	−1	0.5	0.5
176	00001101	11000000	1	2	1	3	1	$1+\epsilon$	0	−1	−1	−1	1	1
177	10001101	01000000	1	2	1	2	0	$1+\epsilon$	0.5	−0.5	−0.5	−1	1	0.5
178	01001101	10000000	1	2	1	3	1	$1+\epsilon$	−0.5	−0.5	−0.5	−1	1	0.5
179	11001101		1	2	1	0	−2						1	
180	00101101	00010000	1	2	−2	−3	−2	$1+\epsilon$	0.5	0.5	−0.5	−1	1	0.5
181	10101101	00010000	1	2	−1	−2	−2	$1+\epsilon$	0.5	0.5	−0.5	−1	1	0.5
182	01101101	00000001	−2	2	−2	3	1	$1+\epsilon$	0.5	0.5	0.5	−1	1	0.5
183	11101101	00010000	0.5	2	−0.5	−1	−1.5	$1+\epsilon$	0.5	0.5	−0.5	−1	0.5	0.5
184	00011101	00100010	0	2	−1	−2	−1	$1+\epsilon$	−1	1	0	−1	1	1
185	10011101		2	2	−1	0	−1						1	
186	01011101	10000000	1	2	0	2	1	$1+\epsilon$	−0.5	−0.5	−0.5	−1	1	0.5
187	11011101		1	2	0	0	−1						1	
188	00111101		1	2	−2	0	1						1	
189	10111101		1	2	−1	0	0						1	
190	01111101	10000000	0.5	2	−0.5	1	0.5	$1+\epsilon$	−0.5	−0.5	−0.5	−1	0.5	0.5
191	11111101		0.5	2	−0.5	0	−0.5						0.5	

TABLE IV
OPTIMAL VALUES OF THE TEMPLATES (PART 4 OF 4).

ID	S 01234567	S_2 01234567	a_{-1}	a_0	a_{+1}	u_0	I	$\hat{a}_0$	d_{-1}	d_0	d_1	$\hat{I}$	δ_1	δ_2
192	00000011	00110000	0	2	0	-3	0	$1+\epsilon$	0	1	-1	-1	2	1
193	10000011	00110000	2	2	2	-3	0	$1+\epsilon$	0	1	-1	-1	2	1
194	01000011	00110000	-2	2	2	-3	0	$1+\epsilon$	0	1	-1	-1	2	1
195	11000011		0	2	3	0	0						2	
196	00100011	00010000	0	2	0	-3	0	$1+\epsilon$	0.5	0.5	-0.5	-1	2	0.5
197	10100011	00010000	1	2	1	-2	0	$1+\epsilon$	0.5	0.5	-0.5	-1	1	0.5
198	01100011		-3	2	3	0	3						2	
199	11100011		-1	2	2	0	1						1	
200	00010011	00100000	0	2	0	-3	0	$1+\epsilon$	-0.5	0.5	-0.5	-1	2	0.5
201	10010011		3	2	3	0	3						2	
202	01010011	00100000	-1	2	1	-2	0	$1+\epsilon$	-0.5	0.5	-0.5	-1	1	0.5
203	11010011		1	2	2	0	1						1	
204	00110011		0	2	0	0	3						2	
205	10110011		1	2	1	0	2						1	
206	01110011		-1	2	1	0	2						1	
207	11110011		0	2	1	0	1						1	
208	00001011	00110000	1	2	-1	-3	-1	$1+\epsilon$	0	1	-1	-1	1	1
209	10001011	01000100	0	2	1	2	1	$1+\epsilon$	1	-1	0	-1	1	1
210	01001011	10000000	1	2	2	3	2	$1+\epsilon$	-0.5	-0.5	-0.5	-1	1	0.5
211	11001011		1	2	2	0	-1						1	
212	00101011	00010000	1	2	-1	-3	-1	$1+\epsilon$	0.5	0.5	-0.5	-1	1	0.5
213	10101011	00010000	1	2	0	-2	-1	$1+\epsilon$	0.5	0.5	-0.5	-1	1	0.5
214	01101011	00001000	-2	2	2	-3	-1	$1+\epsilon$	-0.5	-0.5	0.5	-1	1	0.5
215	11101011	00010000	0.5	2	0.5	-1	-0.5	$1+\epsilon$	0.5	0.5	-0.5	-1	0.5	0.5
216	00011011	00100000	1	2	-1	-2	0	$1+\epsilon$	-0.5	0.5	-0.5	-1	1	0.5
217	10011011		2	2	1	0	1						1	
218	01011011	10000000	1	2	1	2	2	$1+\epsilon$	-0.5	-0.5	-0.5	-1	1	0.5
219	11011011		1	2	1	0	0						1	
220	00111011		1	2	-1	0	2						1	
221	10111011		1	2	0	0	1						1	
222	01111011	10000000	0.5	2	0.5	1	1.5	$1+\epsilon$	-0.5	-0.5	-0.5	-1	0.5	0.5
223	11111011		0.5	2	0.5	0	0.5						0.5	
224	00000111	00110000	-1	2	-1	-3	-1	$1+\epsilon$	0	1	-1	-1	1	1
225	10000111	01000000	-1	2	2	3	2	$1+\epsilon$	0.5	-0.5	-0.5	-1	1	0.5
226	01000111	10001000	0	2	1	2	1	$1+\epsilon$	-1	-1	0	-1	1	1
227	11000111		-1	2	2	0	-1						1	
228	00100111	00010000	-1	2	-1	-2	0	$1+\epsilon$	0.5	0.5	-0.5	-1	1	0.5
229	10100111	01000000	-1	2	1	2	2	$1+\epsilon$	0.5	-0.5	-0.5	-1	1	0.5
230	01100111		-2	2	1	0	1						1	
231	11100111		-1	2	1	0	0						1	
232	00010111	00100000	-1	2	-1	-3	-1	$1+\epsilon$	-0.5	0.5	-0.5	-1	1	0.5
233	10010111	00000100	2	2	2	-3	-1	$1+\epsilon$	0.5	-0.5	0.5	-1	1	0.5
234	01010111	00100000	-1	2	0	-2	-1	$1+\epsilon$	-0.5	0.5	-0.5	-1	1	0.5
235	11010111	00100000	-0.5	2	0.5	-1	-0.5	$1+\epsilon$	-0.5	0.5	-0.5	-1	0.5	0.5
236	00110111		-1	2	-1	0	2						1	
237	10110111	01000000	-0.5	2	0.5	1	1.5	$1+\epsilon$	0.5	-0.5	-0.5	-1	0.5	0.5
238	01110111		-1	2	0	0	1						1	
239	11110111		-0.5	2	0.5	0	0.5						0.5	
240	00001111	11000000	0	2	1	2	1	$1+\epsilon$	0	-1	-1	-1	1	1
241	10001111	01000000	0	2	1	2	1	$1+\epsilon$	0.5	-0.5	-0.5	-1	1	0.5
242	01001111	10000000	0	2	1	2	1	$1+\epsilon$	-0.5	-0.5	-0.5	-1	1	0.5
243	11001111		0	2	1	0	-1						1	
244	00101111	00010000	0	2	-1	-2	-1	$1+\epsilon$	0.5	0.5	-0.5	-1	1	0.5
245	10101111	01000000	-0.5	2	0.5	1.5	1	$1+\epsilon$	0.5	-0.5	-0.5	-1	0.5	0.5
246	01101111	10000000	-0.5	2	0.5	3	2.5	$1+\epsilon$	-0.5	-0.5	-0.5	-1	0.5	0.5
247	11101111		-0.5	2	0.5	0	-0.5						0.5	
248	00011111	00100000	0	2	-1	-2	-1	$1+\epsilon$	-0.5	0.5	-0.5	-1	1	0.5
249	10011111	01000000	0.5	2	0.5	1	0.5	$1+\epsilon$	0.5	-0.5	-0.5	-1	0.5	0.5
250	01011111	10000000	0.5	2	0.5	1.5	1	$1+\epsilon$	-0.5	-0.5	-0.5	-1	0.5	0.5
251	11011111		0.5	2	0.5	0	-0.5						0.5	
252	00111111		0	2	-1	0	1						1	
253	10111111		0.5	2	-0.5	0	0.5						0.5	
254	01111111		-0.5	2	-0.5	0	0.5						0.5	
255	11111111		0	2	0	0	0						1	