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Necessary and Sufficient Condition for a Class of Planar Dynamical Systems Related to CNNs to be Completely Stable

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Abstract—We study global dynamical behavior of cellular neural networks (CNNs) consisting of two cells. Since the output characteristic of each cell is expressed by a piecewise linear function, a CNN with two cells is considered as a planar piecewise linear dynamical system. We present the necessary and sufficient condition for such a CNN to be completely stable under the assumptions that i) self-coupling coefficients take the same value greater than one and ii) biases are set to zero. The condition is explicitly expressed in terms of coupling coefficients between cells.

Index Terms—Cellular neural networks, complete stability, planar dynamical systems

I. INTRODUCTION

In the past two decades, considerable efforts have been devoted to stability analysis of dynamical systems related to recurrent neural networks [1]–[9]. The dynamical behavior of a neural network may be convergent, oscillatory, or even chaotic depending on the values of network parameters such as coupling coefficients between neurons. In many applications of neural networks, it is often required that a neural network is *completely stable*, that is, every trajectory converges to one of the equilibrium points [10]–[12]. Thus it is important from both theoretical and practical point of view to clarify the relationship between the parameters and the complete stability.

In this paper, we study the complete stability of cellular neural networks (CNNs) [13]. A CNN is an analog nonlinear circuit consisting of many signal processing units called cells. The state equation of a CNN with n cells is described by the set of differential equations:

$$\dot{x}_i = -x_i + \sum_{j \in N_i} a_{ij} y_j + b_i, \quad i = 1, 2, \dots, n \quad (1)$$

where x_i represents the state of the i -th cell, y_i the output of the i -th cell which depends on x_i through

$$y_i = f(x_i) \triangleq \frac{1}{2}(|x_i + 1| - |x_i - 1|), \quad (2)$$

a_{ij} the coefficient of the coupling from the j -th cell to the i -th cell, b_i the bias for the i -th cell, and N_i the set of integers

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k such that the k -th cell belongs to the neighborhood of the i -th cell. Since each cell is coupled only with neighboring cells, CNNs are suitable for VLSI implementation. Moreover, due to their rich dynamical behavior, CNNs have found many applications mainly in the field of image processing [14], [15]. As well as other neural network models, it is often required that a CNN is completely stable because the outputs in the steady state is regarded as the result of image processing carried out by the CNN. Many results on the complete stability of CNNs can be found in the literature [16]–[25], but the relationship between the complete stability and the network parameters a_{ij} and b_i has not been clarified yet, even for the simplest case where the CNN consists of only two cells.

The first study of the global dynamical behavior of CNNs consisting of two cells was made by Civalleri and Gilli [18]. They considered CNNs described by

$$\begin{cases} \dot{x}_1 = -x_1 + a_{11}f(x_1) + a_{12}f(x_2) \\ \dot{x}_2 = -x_2 + a_{21}f(x_1) + a_{22}f(x_2) \end{cases} \quad (3)$$

and presented detailed analytical results concerning the location of equilibrium points, the domain of attraction of each equilibrium point, and the complete stability under the assumption that both a_{11} and a_{22} are greater than one. Although most of their results are correct and valuable, it has recently been pointed out that the complete stability analysis for the case where $a_{12}a_{21} < 0$ is not correct [26].

The purpose of this paper is to clarify the complete stability of the planar dynamical system (3) under the assumption that

$$a_{11} = a_{22} > 1 \quad \text{and} \quad a_{12}a_{21} < 0. \quad (4)$$

It is usually assumed for various CNN models that couplings between cells are space-invariant [15]. The first condition in (4) corresponds to this assumption. Exploring the properties of the phase portrait of the system in detail, we will derive the necessary and sufficient condition for (3) satisfying (4) to be completely stable. Since the case where $a_{12}a_{21} \geq 0$ has already been solved by Civalleri and Gilli [18], the results of this paper completes the complete stability analysis for (3) with $a_{11} = a_{22} > 1$.

The technique used for the stability analysis in this paper is restricted to two-cell CNNs and cannot be directly applied to more general CNNs. However, the results themselves are important for the following two reasons. First, stability analysis of a two-cell CNN plays an important role for that of an n -cell CNN in some cases (for example, see Theorem 4 in [27] and [28]). Second, since the system (3) satisfying (4) is

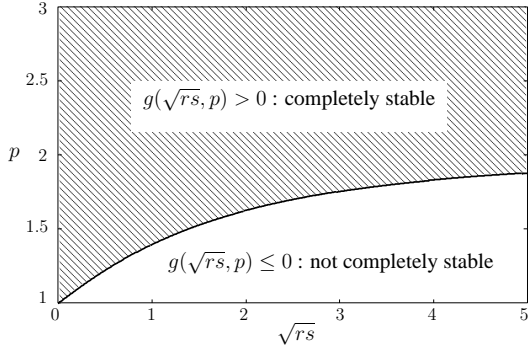


Fig. 1. Parameter region for $g(\sqrt{rs}, p) > 0$.

a special case of CNNs with the opposite-sign template, the results given in this paper can be regarded as a fundamental work for the complete stability analysis of such CNNs. In particular, if we try to derive a condition for a CNN with the opposite-sign template to be completely stable independent of the number of cells, it suffices for us to consider only parameter values satisfying the complete stability condition given in this paper.

II. MAIN RESULTS

Let us consider the dynamical system (3) under the assumption (4). Since we can assume without loss of generality that $0 < a_{12} \leq -a_{21}$, the system can be expressed in terms of three parameters p , r and s as follows:

$$\begin{cases} \dot{x}_1 = -x_1 + pf(x_1) + sf(x_2) \\ \dot{x}_2 = -x_2 - rf(x_1) + pf(x_2) \end{cases} \quad (p > 1, r \geq s > 0) \quad (5)$$

where $f(\cdot)$ is a piecewise linear function defined by (2). In the following, we denote the solution of the system (5) passing through a point $P \in \mathbb{R}^2$ at $t = 0$ by $\psi(t, P) = (\psi_1(t, P), \psi_2(t, P))$. The system (5) is said to be completely stable if for any P the solution $\psi(t, P)$ converges to some equilibrium point as t goes to infinity.

For the system (5), we define the function $g(\sqrt{rs}, p)$ as

$$g(\sqrt{rs}, p) \triangleq \exp\left(\frac{2(p-1)}{\sqrt{rs}} \arctan\left(\frac{p}{\sqrt{rs}}\right)\right) - \frac{rs}{(p-1)^2(p^2 + rs)} \quad (6)$$

where $\arctan(\cdot)$ takes its principal value, that is, $-\pi/2 < \arctan(x) < \pi/2$ for all $x \in \mathbb{R}$. This function plays an important role in our stability analysis.

The following theorem is the main result of this paper.

Theorem 1: The system (5) is completely stable if and only if $p - 1 \geq s$ and $g(\sqrt{rs}, p) > 0$ hold simultaneously.

The second condition specifies the relationship between the self-coupling coefficient p and the geometric average of mutual coupling coefficients r and s . By solving the equation $g(\sqrt{rs}, p) = 0$ for p numerically for each value of \sqrt{rs} , we can draw the curve $g(\sqrt{rs}, p) = 0$ as shown in Fig.1. The \sqrt{rs} - p space is divided into two regions by the curve $g(\sqrt{rs}, p) = 0$; the system (5) is completely stable for the upper region but

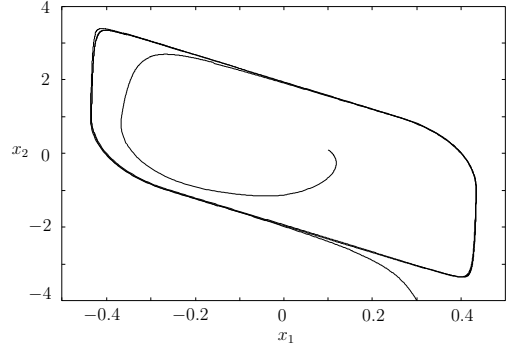


Fig. 2. Two trajectories starting at $(0.1, 0.1)$ and $(0.3, -4.0)$ for the CNN considered in Example 1.

not for the lower one including the curve $g(\sqrt{rs}, p) = 0$. Note that for any fixed value of \sqrt{rs} , the equation $g(\sqrt{rs}, p) = 0$ has a unique solution since $g(\sqrt{rs}, p)$ is monotone increasing in p . Also, it is easily seen that the value of p satisfying $g(\sqrt{rs}, p) = 0$ is monotone increasing in \sqrt{rs} , and less than 2 for any value of \sqrt{rs} . We thus can derive a much simpler complete stability condition from Theorem 1 as follows:

Corollary 1: The system (5) is completely stable if $p - 1 \geq s$ and $p \geq 2$ hold simultaneously.

By combining Theorem 1 with the results given in [18], we have the following theorem.

Theorem 2: Under the assumption that $a_{11} = a_{22} > 1$, the system (3) is completely stable if and only if one of the following two conditions holds.

- 1) $a_{12}a_{21} \geq 0$
- 2) $a_{12}a_{21} < 0$, $a_{11} - 1 \geq \min\{|a_{12}|, |a_{21}|\}$ and $g(\sqrt{-a_{12}a_{21}}, a_{11}) > 0$

In order to show validity of the main result of this paper, we now give two illustrative examples.

Example 1: Let us consider a two-cell CNN with the parameters given by $a_{11} = a_{22} = 1.36$, $a_{12} = 0.16$ and $a_{21} = -5.0$. For these parameter values, we have $g(\sqrt{-a_{12}a_{21}}, a_{11}) = g(\sqrt{0.8}, 1.36) \simeq -0.113 < 0$. Hence it follows from Theorem 2 that this CNN is not completely stable. This is verified by solving the state equation numerically for some appropriate initial conditions. In fact, as shown in Fig.2, two trajectories starting at $(0.1, 0.1)$ and $(0.3, -4.0)$ converge to the same limit cycle.

Example 2: Let us increase the self-coupling coefficients of the CNN in Example 1 from 1.36 to 1.37. In this case we have $g(\sqrt{-a_{12}a_{21}}, a_{11}) = g(\sqrt{0.8}, 1.37) \simeq 0.090 > 0$. So it follows from Theorem 2 that this CNN is completely stable. In fact, as shown in Fig.3, two trajectories starting at the same points as Example 1, i.e., $(0.1, 0.1)$ and $(0.3, -4.0)$, converge to two distinct equilibrium points at $(1.21, -6.37)$ and $(-1.21, 6.37)$.

III. PROOF OF THEOREM 1

We first show some known results on the complete stability of the system (5).

Lemma 1: If the system (5) satisfies $p - 1 < s$ then it is not completely stable.

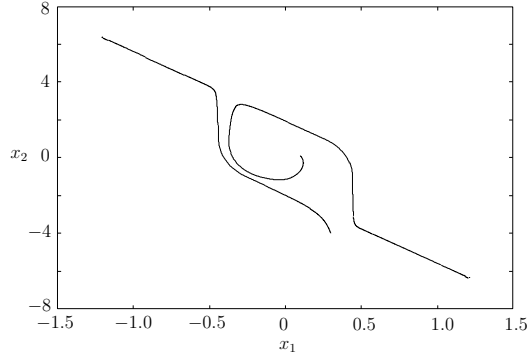


Fig. 3. Two trajectories starting at $(0.1, 0.1)$ and $(0.3, -4.0)$ for the CNN considered in Example 2.

Proof: See [18].

Lemma 2: If the system (5) satisfies $p - 1 \geq \sqrt{rs}$ then it is completely stable.

Proof: See [26].

From these two lemmas, we can hereafter concentrate our attention on the case where

$$s \leq p - 1 < \sqrt{rs}. \quad (7)$$

Note that this condition implies $r > s$.

The phase portrait of the system (5) with (7) is shown in Fig.4 where nullclines, equilibrium points and directions of the flow at some points are drawn. Let

$$R^{(i,j)} \triangleq \{(x_1, x_2) \mid x_1 \in J^{(i)}, x_2 \in J^{(j)}\}, \quad i, j \in \{-, 0, +\}$$

be nine regions in the state space of the system (5) where $J^{(-)} = (-\infty, -1)$, $J^{(0)} = [-1, 1]$ and $J^{(+)} = (1, \infty)$. If $s < p - 1$, there are five equilibrium points O , $E^{(0,+)} = (-s/(p-1), p + rs/(p-1))$, $E^{(-,+)} = (-p + s, p + r)$, $E^{(0,-)} = (s/(p-1), -p - rs/(p-1))$ and $E^{(+,-)} = (p - s, -p - r)$ where O is an unstable focus, $E^{(0,+)}$ and $E^{(0,-)}$ are saddle points, and $E^{(-,+)}$ and $E^{(+,-)}$ are stable nodes. If $p - 1 = s$, on the other hand, there are only three equilibrium points O , $E^{(0,+)}$ and $E^{(0,-)}$. In this case, $E^{(0,+)}$ and $E^{(0,-)}$ are unstable but they are not saddle points. Four breaking points of the nullclines on the boundary of $R^{(0,0)}$ are expressed by $A = (-s/(p-1), 1)$, $B = ((p-1)/r, 1)$, $C = (s/(p-1), -1)$ and $D = (-(p-1)/r, -1)$.

It is easily seen from Fig.4 that for any $P \in \mathbb{R}^2$ if the trajectory $\psi(t, P)$ once satisfies $|\psi_1(t, P)| > s/(p-1)$, it necessarily converges to one of the equilibrium points. Thus the system (5) satisfying (7) is not completely stable if and only if there exists a trajectory which stays in the region $\{(x_1, x_2) \mid |x_1| \leq s/(p-1)\}$ and never converges.

In the previous work [26], the authors investigated the behavior of the unstable manifold of $E^{(0,+)}$ and showed that it touches the line segment OC if and only if $g(\sqrt{rs}, p) \leq 0$. In particular, it passes through the point C , which implies that there exists a heteroclinic orbit connecting $E^{(0,+)}$ and $E^{(0,-)}$, if and only if $g(\sqrt{rs}, p) = 0$. From these results and the Poincaré-Bendixon theorem, the following lemma is derived.

Lemma 3: If the system (5) satisfies (7) and $g(\sqrt{rs}, p) \leq 0$ then it is not completely stable.

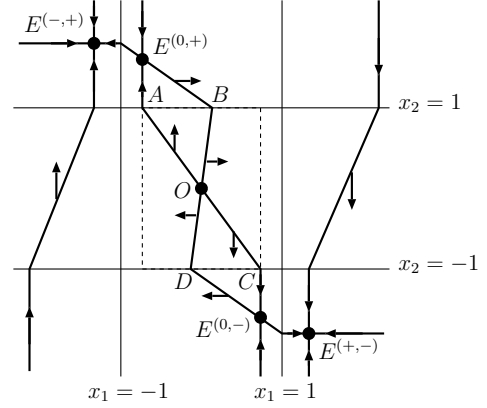


Fig. 4. Phase portrait of the system (5) with (7).

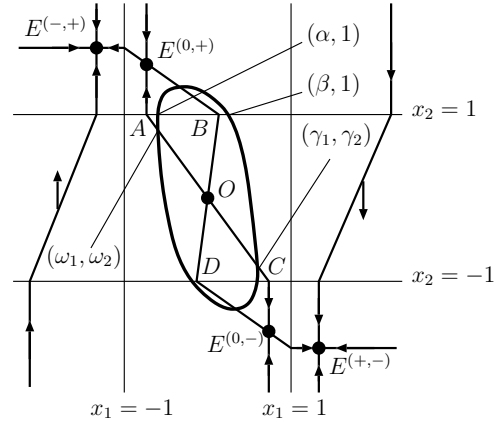


Fig. 5. Possible limit cycle.

Proof: See [26].

According to Lemmas 1–3, the final step to prove Theorem 1 is to show that the system (5) is completely stable if (7) and $g(\sqrt{rs}, p) > 0$ hold. Since the heteroclinic orbit connecting $E^{(0,+)}$ and $E^{(0,-)}$ does not exist in this case, it suffices for us to show that the system (5) satisfying (7) and $g(\sqrt{rs}, p) > 0$ does not have a limit cycle passing through the line segments OA , AB , $E^{(0,+)}B$, OC , CD , $E^{(0,-)}D$ and OD consecutively (see Fig.5). The proof will be done by contradiction, that is, we first assume there exists such a limit cycle and then show this leads to a contradiction. Let $(\alpha, 1)$ and $(\beta, 1)$ be the intersections of the limit cycle with the line segment AB , and the line segment connecting B and $(1, 1)$, respectively. Then α and β satisfy $-s/(p-1) < \alpha < (p-1)/r < \beta$. In addition, the following four lemmas hold.

Lemma 4: Assume that the system (5) satisfying (7) and $g(\sqrt{rs}, p) > 0$ has a limit cycle shown in Fig.5. Then $\beta < s + p(p-1)/r$.

Proof: Suppose first that $s + p(p-1)/r > 1$. Then $\beta < s + p(p-1)/r$ apparently holds because β must be at least smaller than 1 in order for $\psi(t, (\beta, 1))$ to cross the line segment OC . Suppose next that $s + p(p-1)/r \leq 1$. In this case, it was

shown in [26] that the point $(s + p(p-1)/r, 1)$ is on the unstable manifold of $E^{(0,+)}$. Since two different trajectories never intersect, β must be smaller than $s + p(p-1)/r$. ■

Lemma 5: Assume that the system (5) satisfying (7) and $g(\sqrt{rs}, p) > 0$ has a limit cycle shown in Fig.5. Then $h(\alpha, \beta) = 0$ holds where

$$h(\alpha, \beta) \triangleq \left(\alpha + \frac{s}{p-1}\right) \left\{s + \frac{p(p-1)}{r} - \alpha\right\}^{p-1} - \left(\beta + \frac{s}{p-1}\right) \left\{s + \frac{p(p-1)}{r} - \beta\right\}^{p-1}. \quad (8)$$

Proof: Let t_1 be the smallest positive value of t for which $\psi(t, (\alpha, 1))$ is on the line segment $BE^{(0,+)}$. By solving the differential equation (5) for the region $R^{(0,+)}$ with the initial condition $(x_1(0), x_2(0)) = (\alpha, 1)$, we derive the explicit formula for the trajectory $\psi(t, (\alpha, 1))$ ($0 \leq t \leq t_1$) as follows:

$$\begin{aligned} \psi_1(t, (\alpha, 1)) &= \left(\alpha + \frac{s}{p-1}\right) e^{(p-1)t} - \frac{s}{p-1} \\ \psi_2(t, (\alpha, 1)) &= \frac{r}{p} \left\{\alpha - s - \frac{p(p-1)}{r}\right\} e^{-t} \\ &\quad - \frac{r}{p} \left(\alpha + \frac{s}{p-1}\right) e^{(p-1)t} + p + \frac{rs}{p-1} \end{aligned} \quad (9)$$

Since $\dot{\psi}_2(t, (\alpha, 1))$ vanishes at $t = t_1$, we have

$$t_1 = \frac{1}{p} \log \left(\frac{s + \frac{p(p-1)}{r} - \alpha}{\alpha(p-1) + s} \right).$$

Thus $\psi_1(t_1, (\alpha, 1))$ is expressed in terms of α as

$$\begin{aligned} \psi_1(t_1, (\alpha, 1)) &= \left(\alpha + \frac{s}{p-1}\right) \left\{ \frac{s + \frac{p(p-1)}{r} - \alpha}{(p-1)\alpha + s} \right\}^{(p-1)/p} - \frac{s}{p-1}. \end{aligned}$$

Next let t_2 be the largest negative value of t for which $\psi(t, (\beta, 1))$ is on the line segment $BE^{(0,+)}$. Then, in a similar way as above, we can express $\psi_1(t_2, (\beta, 1))$ in terms of β as

$$\begin{aligned} \psi_1(t_2, (\beta, 1)) &= \left(\beta + \frac{s}{p-1}\right) \left\{ \frac{s + \frac{p(p-1)}{r} - \beta}{(p-1)\beta + s} \right\}^{(p-1)/p} - \frac{s}{p-1}. \end{aligned}$$

Since $\psi_1(t_1, (\alpha, 1)) = \psi_1(t_2, (\beta, 1))$, we have $h(\alpha, \beta) = 0$. ■

Lemma 6: Assume that the system (5) satisfying (7) and $g(\sqrt{rs}, p) > 0$ has a limit cycle shown in Fig.5. If $p \geq 2$ then $\beta^2 > \alpha^2$ holds.

Proof: Let $l_1(u, v) \triangleq v^2 - u^2$. We shall show that $l_1(u, v)$ is positive for all u and v satisfying

$$-\frac{s}{p-1} < u < \frac{p-1}{r} < v < s + \frac{p(p-1)}{r} \quad (10)$$

and $h(u, v) = 0$. Note that $l_1(u, v)$ is intrinsically a function of u only because v is defined by u implicitly through $h(u, v) = 0$. Since $v \rightarrow s + p(p-1)/r$ as $u \rightarrow -s/(p-1)$, we have

$$\lim_{u \rightarrow -s/(p-1)} l_1(u, v) = \left\{s + \frac{p(p-1)}{r}\right\}^2 - \left(-\frac{s}{p-1}\right)^2 > 0 \quad (11)$$

Also, since $v \rightarrow (p-1)/r$ as $u \rightarrow (p-1)/r$, we have

$$\lim_{u \rightarrow (p-1)/r} l_1(u, v) = 0. \quad (12)$$

Making use of the implicit function theorem and the assumption $p \geq 2$, we have

$$\begin{aligned} \frac{dl_1(u, v)}{du} &= 2v \cdot \frac{dv}{du} - 2u \\ &= 2v \cdot \left(-\frac{\partial h / \partial u}{\partial h / \partial v}\right) - 2u \\ &= -2v \cdot \frac{\left(\frac{p-1}{r} - u\right) \left\{-u + s + \frac{p(p-1)}{r}\right\}^{p-2}}{\left(v - \frac{p-1}{r}\right) \left\{-v + s + \frac{p(p-1)}{r}\right\}^{p-2}} - 2u \\ &\leq -2v \cdot \frac{\frac{p-1}{r} - u}{v - \frac{p-1}{r}} - 2u \\ &= -\frac{2(p-1)}{r} \cdot \frac{v - u}{v - \frac{p-1}{r}} \\ &< 0. \end{aligned} \quad (13)$$

From (11)–(13), we can conclude that $l_1(u, v) > 0$ for all u and v satisfying (10) and $h(u, v) = 0$ if $p \geq 2$. ■

Lemma 7: Assume that the system (5) satisfying (7) and $g(\sqrt{rs}, p) > 0$ has a limit cycle shown in Fig.5. If $p < 2$ then $\beta \geq (p-1)(p/r - \alpha)$ holds.

Proof: Let $l_2(u, v) \triangleq v - (p-1)(p/r - u)$. We shall show in the following that $l_2(u, v)$ is nonnegative for all u and v satisfying (10) and $h(u, v) = 0$. Since $v \rightarrow (p-1)/r$ as $u \rightarrow (p-1)/r$, we have

$$\lim_{u \rightarrow (p-1)/r} l_2(u, v) = \frac{p-1}{r} - (p-1) \left(\frac{p}{r} - \frac{p-1}{r}\right) = 0. \quad (14)$$

Let us consider the value of

$$\lim_{u \rightarrow (p-1)/r} \frac{dl_2(u, v)}{du} = (p-1) + \lim_{u \rightarrow (p-1)/r} \frac{dv}{du}.$$

Applying the implicit function theorem and the De L'Hôpital's theorem to the second term of the right-hand side, we have

$$\begin{aligned} \lim_{u \rightarrow (p-1)/r} \frac{dv}{du} &= - \lim_{u \rightarrow (p-1)/r} \frac{\partial h / \partial u}{\partial h / \partial v} \\ &= - \lim_{u \rightarrow (p-1)/r} \frac{\left(\frac{p-1}{r} - u\right) \left\{-u + s + \frac{p(p-1)}{r}\right\}^{p-2}}{\left(v - \frac{p-1}{r}\right) \left\{-v + s + \frac{p(p-1)}{r}\right\}^{p-2}} \\ &= - \lim_{u \rightarrow (p-1)/r} \frac{\frac{p-1}{r} - u}{v - \frac{p-1}{r}} \\ &= \lim_{u \rightarrow (p-1)/r} \frac{1}{dv/du} \end{aligned}$$

which implies $\lim_{u \rightarrow (p-1)/r} dv/du = -1$ because $dv/du < 0$ for all u satisfying (10) and $h(u, v) = 0$. Hence we have

$$\lim_{u \rightarrow (p-1)/r} \frac{dl_2(u, v)}{du} = (p-1) - 1 = p-2 < 0. \quad (15)$$

It follows from (14) and (15) that there exists a positive number ϵ such that $l_2(u, v) > 0$ for all u satisfying $(p-1)/r - \epsilon < u < (p-1)/r$. Assume now that $l_2(u, v) < 0$ for

some u satisfying $-s/(p-1) < u \leq (p-1)/r - \epsilon$. Then there must exist a u such that

$$h(u, v) = 0, \quad l_2(u, v) = 0 \quad \text{and} \quad \frac{dl_2(u, v)}{du} \geq 0. \quad (16)$$

However, we can show that there is no u satisfying (16) as follows. Taking (8) into account, we have

$$\begin{aligned} & \left. \frac{dl_2(u, v)}{du} \right|_{h(u, v)=0} \\ &= (p-1) - \frac{\left(\frac{p-1}{r} - u\right) \left(v + \frac{s}{p-1}\right) \left\{-v + s + \frac{p(p-1)}{r}\right\}}{\left(v - \frac{p-1}{r}\right) \left(u + \frac{s}{p-1}\right) \left\{-u + s + \frac{p(p-1)}{r}\right\}}. \end{aligned}$$

Substituting $v = (p-1)(p/r - u)$ into the right-hand side of the above equation, we have

$$\begin{aligned} & \left. \frac{dl_2(u, v)}{du} \right|_{h(u, v)=0, l_2(u, v)=0} \\ &= -(p-1) \left\{ \frac{p(2-p)}{r(p-1)^2} \cdot \frac{(p-1)^2 + rs}{-u + s + \frac{p(p-1)}{r}} \right\}. \end{aligned}$$

Since the right-hand side is negative if $p < 2$ and $-s/(p-1) < u < (p-1)/r$, there is no u which satisfies (16). This is a contradiction. ■

Let (γ_1, γ_2) and (ω_1, ω_2) be the intersections of the possible limit cycle with the line segment OC and OA , respectively (See Fig.5). Then, since the flow is symmetric with respect to the origin, these two intersections must satisfy

$$(\omega_1, \omega_2) = (-\gamma_1, -\gamma_2). \quad (17)$$

By using the same technique as in the proof of Lemma 5, we can derive the following lemma.

Lemma 8: Assume that the system (5) satisfying (7) and $g(\sqrt{rs}, p) > 0$ has a limit cycle shown in Fig.5. Then γ_2 and ω_2 are expressed as $\gamma_2 = -m(\beta)$ and $\omega_2 = m(\alpha)$ where

$$\begin{aligned} m(u) &\triangleq \frac{(p-1)\sqrt{s+ru^2}}{\sqrt{s\{(p-1)^2+rs\}}} \\ &\times \exp\left(\frac{p-1}{\sqrt{rs}} \arctan\left(\frac{p-1}{\sqrt{rs}} \cdot \frac{u + \frac{s}{p-1}}{u - \frac{p-1}{r}}\right)\right). \quad (18) \end{aligned}$$

Proof: Let us consider the trajectory $\psi(t, (\beta, 1)) = (\psi_1(t, (\beta, 1)), \psi_2(t, (\beta, 1)))$. Let t_1 be the smallest positive value of t satisfying $\psi_1(t, (\beta, 1)) = 0$. Then γ_2 is expressed by $\gamma_2 = \psi_2(t_1, (\beta, 1))$. By solving the differential equation (5) for the region $R^{(0,0)}$ under the initial condition $(x_1(0), x_2(0)) = (\beta, 1)$, we can obtain the explicit formulae for $\psi_1(t, (\beta, 1))$ and $\psi_2(t, (\beta, 1))$ as follows:

$$\psi_1(t, (\beta, 1)) = e^{(p-1)t} \left(\beta \cos(\sqrt{rs}t) + \sqrt{\frac{s}{r}} \sin(\sqrt{rs}t) \right) \quad (19)$$

$$\psi_2(t, (\beta, 1)) = e^{(p-1)t} \left(\cos(\sqrt{rs}t) - \beta \sqrt{\frac{r}{s}} \sin(\sqrt{rs}t) \right) \quad (20)$$

Differentiating the right-hand side of (19) with respect to t and setting it to 0, we have

$$\begin{aligned} & \{(p-1)\beta + s\} \cos(\sqrt{rs}t) \\ & + \left\{ (p-1) \sqrt{\frac{s}{r}} - \beta \sqrt{rs} \right\} \sin(\sqrt{rs}t) = 0. \end{aligned}$$

Solving this equation for t , we have the explicit formula for t_1 as

$$t_1 = \frac{1}{\sqrt{rs}} \arctan\left(\frac{p-1}{\sqrt{rs}} \cdot \frac{\beta + \frac{s}{p-1}}{\beta - \frac{p-1}{r}}\right) \quad \left(0 < t_1 < \frac{\pi}{2\sqrt{rs}}\right).$$

Substituting $t = t_1$ into (20) and simplifying the formula, we derive

$$\begin{aligned} \psi_2(t_1, (\beta, 1)) &= -\frac{(p-1)\sqrt{s+r\beta^2}}{\sqrt{s\{(p-1)^2+rs\}}} \\ &\times \exp\left(\frac{p-1}{\sqrt{rs}} \arctan\left(\frac{p-1}{\sqrt{rs}} \cdot \frac{\beta + \frac{s}{p-1}}{\beta - \frac{p-1}{r}}\right)\right) \end{aligned}$$

which completes our proof for $\gamma_2 = -m(\beta)$. The second equation $\omega_2 = m(\alpha)$ is derived in a similar way. ■

By analyzing the function $m(u)$ defined by (18) itself, we obtain the following two lemmas.

Lemma 9: The function $m(u)$ is monotone increasing for $u > (p-1)/r$.

Proof: We will show the monotonicity of $m^2(u)$ instead of $m(u)$ itself. Differentiating $m^2(u)$, we have

$$\begin{aligned} \frac{dm^2(u)}{du} &= \tilde{m}(u) \left\{ 2ru + (s+ru^2) \cdot \frac{2(p-1)}{\sqrt{rs}} \right. \\ &\quad \left. \times \frac{d}{du} \arctan\left(\frac{p-1}{\sqrt{rs}} \cdot \frac{u + \frac{s}{p-1}}{u - \frac{p-1}{r}}\right) \right\} \quad (21) \end{aligned}$$

where $\tilde{m}(u)$ is defined by

$$\begin{aligned} \tilde{m}(u) &\triangleq \frac{(p-1)^2}{s\{(p-1)^2+rs\}} \\ &\times \exp\left(\frac{2(p-1)}{\sqrt{rs}} \arctan\left(\frac{p-1}{\sqrt{rs}} \cdot \frac{u + \frac{s}{p-1}}{u - \frac{p-1}{r}}\right)\right). \quad (22) \end{aligned}$$

It is apparent that $\tilde{m}(u) > 0$ for $u > (p-1)/r$. Taking $\arctan'(x) = 1/(x^2+1)$ into account, we have

$$\frac{d}{du} \arctan\left(\frac{p-1}{\sqrt{rs}} \cdot \frac{u + \frac{s}{p-1}}{u - \frac{p-1}{r}}\right) = -\frac{\sqrt{rs}}{s+ru^2}. \quad (23)$$

Substituting (23) into (21), we have $dm^2(u)/du = 2r\tilde{m}(u)(u - (p-1)/r)$ which is positive for $u > (p-1)/r$. ■

Lemma 10: If $p < 2$ and $g(\sqrt{rs}, p) > 0$ then $m^2((p-1)(p/r - u)) > m^2(u)$ for any u satisfying $-s/(p-1) < u < (p-1)/r$.

Proof: Let $o(u) \triangleq m^2((p-1)(p/r - u)) - m^2(u)$. Since $(p-1)(p/r - u) \rightarrow s + p(p-1)/r$ as $u \rightarrow -s/(p-1)$, we have

$$\begin{aligned} & \lim_{u \rightarrow -s/(p-1)} o(u) \\ &= m^2\left(s + \frac{p(p-1)}{r}\right) - m^2\left(-\frac{s}{p-1}\right) \\ &= \frac{(p-1)^2(p^2+rs)}{rs} \exp\left(\frac{2(p-1)}{\sqrt{rs}} \arctan\left(\frac{p}{\sqrt{rs}}\right)\right) - 1 \\ &> 0 \end{aligned}$$

where the last inequality holds from $g(\sqrt{rs}, p) > 0$. Also, since $(p-1)(p/r - u) \rightarrow (p-1)/r$ as $u \rightarrow (p-1)/r$, we have

$$\begin{aligned} & \lim_{u \rightarrow (p-1)/r} o(u) \\ &= \lim_{u \rightarrow (p-1)/r+0} m^2(u) - \lim_{u \rightarrow (p-1)/r-0} m^2(u) \\ &= \frac{(p-1)^2}{rs} \exp\left(\frac{2(p-1)}{\sqrt{rs}} \cdot \frac{\pi}{2}\right) \\ &\quad - \frac{(p-1)^2}{rs} \exp\left(\frac{2(p-1)}{\sqrt{rs}} \cdot \left(-\frac{\pi}{2}\right)\right) \\ &> 0 \end{aligned}$$

where $u \rightarrow (p-1)/r+0$ and $u \rightarrow (p-1)/r-0$ mean that u approaches to $(p-1)/r$ from above and below, respectively. From the above two inequalities, it suffices for us to show that for any u satisfying $o'(u) = 0$ the function $o(u)$ takes a positive value. Differentiating $o(u)$, we have

$$\begin{aligned} \frac{do(u)}{du} &= \frac{d}{du} m^2\left((p-1)\left(\frac{p}{r} - u\right)\right) - \frac{d}{du} m^2(u) \\ &= 2r\tilde{m}\left((p-1)\left(\frac{p}{r} - u\right)\right) \\ &\quad \times \left\{ (p-1)\left(\frac{p}{r} - u\right) - \frac{p-1}{r} \right\} \{-(p-1)\} \\ &\quad - 2r\tilde{m}(u)\left(u - \frac{p-1}{r}\right) \\ &= 2r\left(u - \frac{p-1}{r}\right) \\ &\quad \times \left\{ (p-1)^2\tilde{m}\left((p-1)\left(\frac{p}{r} - u\right)\right) - \tilde{m}(u) \right\} \end{aligned}$$

where $\tilde{m}(u)$ is given by (22). Thus $o'(u) = 0$ if and only if $\tilde{m}\left((p-1)(p/r - u)\right) = \tilde{m}(u)/(p-1)^2$. We therefore have

$$\begin{aligned} o(u)|_{o'(u)=0} &= \tilde{m}\left((p-1)\left(\frac{p}{r} - u\right)\right) \left\{ s + r(p-1)^2\left(\frac{p}{r} - u\right)^2 \right\} \\ &\quad - \tilde{m}(u)(s + ru^2) \\ &= \frac{\tilde{m}(u)}{(p-1)^2} \left\{ s + r(p-1)^2\left(\frac{p}{r} - u\right)^2 \right\} - \tilde{m}(u)(s + ru^2) \\ &= p\tilde{m}(u) \left\{ \frac{s(2-p)}{(p-1)^2} + \frac{p}{r} - 2u \right\} \\ &> p\tilde{m}(u) \left\{ \frac{s(2-p)}{(p-1)^2} + \frac{p}{r} - \frac{2(p-1)}{r} \right\} \\ &= p(2-p)\tilde{m}(u) \left\{ \frac{1}{r} + \frac{s}{(p-1)^2} \right\} \\ &> 0 \end{aligned}$$

where the last inequality holds from $p < 2$. ■

Now we are ready for presenting the following lemma which completes our proof of Theorem 1.

Lemma 11: If the system (5) satisfies (7) and $g(\sqrt{rs}, p) > 0$ then it has no limit cycle.

Proof: Suppose there exists a limit cycle shown in Fig.5. In the case where $p \geq 2$, as shown in Lemma 6, $\alpha < (p-1)/r < \beta$ and $\alpha^2 < \beta^2$ hold. One can easily see from these

conditions that $m^2(\alpha) < m^2(\beta)$. In the case where $p < 2$ and $g(\sqrt{rs}, p) > 0$, we also have from Lemmas 7, 9 and 10 that

$$m^2(\alpha) < m^2\left((p-1)\left(\frac{p}{r} - \alpha\right)\right) < m^2(\beta).$$

Note that $m^2(\alpha) < m^2(\beta)$ implies $\omega_2 < -\gamma_2$. However, this contradicts (17). Thus there is no limit cycle if either i) $p \geq 2$ or ii) $p < 2$ and $g(\sqrt{rs}, p) > 0$ holds. Since $p \geq 2$ implies $g(\sqrt{rs}, p) > 0$, these two conditions can be unified as $g(\sqrt{rs}, p) > 0$. ■

IV. EXTENSION TO FULL-RANGE CNN MODEL

Corinto and Gilli [29] studied the complete stability of a more general CNN model described by

$$\begin{cases} \dot{x}_1 = -(1+\mu)x_1 + (a_{11} + \mu)f(x_1) + a_{12}f(x_2) \\ \dot{x}_2 = -(1+\mu)x_2 + a_{21}f(x_1) + (a_{22} + \mu)f(x_2) \end{cases} \quad (24)$$

where $\mu \geq 0$ is an additional parameter. This is a special case of the so-called full-range CNNs (FRCNNs). By extending some results given in [26], they have shown that the dynamical behavior of the system (24) is not equivalent to that of (3) for some $\mu \geq 0$. In addition, they have given the following theorem.

Theorem 3: If the parameters in (24) satisfy

$$a_{11} = a_{22} = p, \quad a_{12} = s, \quad a_{21} = -r, \quad 0 < s \leq p-1 < r \quad (25)$$

and

$$\exp\left(\frac{2(p-1)}{\sqrt{rs}} \arctan\left(\frac{p}{\sqrt{rs}}\right)\right) - \frac{rs(1+\mu)^2}{(p-1)^2\{(p+\mu)^2 + rs\}} > 0 \quad (26)$$

then the system (24) is completely stable.

It is easily seen that Theorem 3 is a generalization of Theorem 1. In order to prove Theorem 3, Corinto and Gilli first showed that if

$$\exp\left(\frac{2(p-1)}{\sqrt{rs}} \cdot \frac{\pi}{2}\right) - \frac{rs}{(p-1)^2} > 0 \quad (27)$$

then the system (24) with (25) is completely stable. They next considered two points $(\sqrt{r_1 s_1}, p_1)$ and $(\sqrt{r_1 s_1}, p_2)$ in the $\sqrt{rs} - p$ plane such that the former satisfies both (26) and (27) and the latter satisfies only (26), and claimed that the system (24) never presents a structural instability of the first degree at any point on the line segment connecting $(\sqrt{r_1 s_1}, p_1)$ and $(\sqrt{r_1 s_1}, p_2)$. In particular, they claimed that a limit cycle of multiplicity 2 can never appear because there is no limit cycle at $(\sqrt{r_1 s_1}, p_1)$. However, this is not correct because a limit cycle of multiplicity two, or a half-stable limit cycle, may suddenly appear in the process of decreasing the value of p from p_1 to p_2 . Therefore, we have to say that the proof for Theorem 3 given in [29] is incomplete.

On the other hand, however, Theorem 3 can be rigorously proved by extending the results given in the previous section. Let us suppose that the system (24) satisfying (25) has a limit cycle shown in Fig.5¹. First, taking into account that the state

¹Expressions of the equilibrium points must be modified as $E^{(0,+)} = (-s/(p-1), (rs/(p-1) + p + \mu)/(1 + \mu))$, and so on.

equations (3) and (24) have the same form in the region $R^{(0,0)}$, we can easily see that Lemmas 8 and 9 hold for the system (24) satisfying (25). Second, we can generalize Lemmas 4–7 and 10 for the system (24) with (25) as follows:

Lemma 12: $\beta < s/(1 + \mu) + (p + \mu)(p - 1)/\{r(1 + \mu)\}$.

Lemma 13: α and β satisfy

$$\begin{aligned} & \left(\alpha + \frac{s}{p-1}\right)^{\mu+1} \left\{s + \frac{(p+\mu)(p-1)}{r} - \alpha(\mu+1)\right\}^{p-1} \\ &= \left(\beta + \frac{s}{p-1}\right)^{\mu+1} \left\{s + \frac{(p+\mu)(p-1)}{r} - \beta(\mu+1)\right\}^{p-1}. \end{aligned}$$

Lemma 14: If $p \geq \mu + 2$ then $\beta^2 > \alpha^2$.

Lemma 15: If $p < \mu + 2$ then $\beta \geq ((p-1)/(\mu+1))((p+\mu)/r - \alpha)$.

Lemma 16: If $p < \mu + 2$ and (26) hold then $m^2((p-1)((p+\mu)/r - u)/(\mu+1)) > m^2(u)$ for any u satisfying $-s/(p-1) < u < (p-1)/r$.

We omit the proofs of these five lemmas because they are similar to those of Lemmas 4–7 and 10. Theorem 3 can be proved by using Lemmas 8, 9 and 12–16 in the same way as the proof of Lemma 11. In addition, since Lemmas 1 and 2 hold for the system (24) with (25) and Lemma 3 can be generalized² in the same way as above, we have the following theorem which is a generalization of Theorem 1.

Theorem 4: The system (24) with (25) is completely stable if and only if $p - 1 \geq s$ and (26) hold simultaneously.

V. CONCLUSION

We have studied the complete stability of planar piecewise linear dynamical systems related to CNNs consisting of two cells. Exploring the phase portrait in detail, we have derived the necessary and sufficient condition for such a system to be completely stable. Complete stability analysis of the system for more general cases, for example, the case where biases are set to nonzero values or the case where self-coupling coefficients do not take the same value, is the future problem.

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²Generalized version of Lemma 3 is given in [29].