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Maximizing Algebraic Connectivity in the Space of Graphs with Fixed Number of Vertices and Edges

Kohnosuke Ogiwara, Tatsuya Fukami and Norikazu Takahashi, *Member, IEEE*

Abstract—The second smallest eigenvalue of the Laplacian matrix, also known as the algebraic connectivity, characterizes the performance of some dynamic processes on networks, such as consensus in multiagent networks, synchronization of coupled oscillators, random walks on graphs, and so on. In a multiagent network, for example, the larger the algebraic connectivity of the graph representing interactions between agents is, the faster the convergence speed of a representative consensus algorithm is. This paper tackles the problem of finding graphs that maximize or locally maximize the algebraic connectivity in the space of graphs with a fixed number of vertices and edges. It is shown that some well-known classes of graphs such as star graphs, cycle graphs, complete bipartite graphs and circulant graphs are algebraic connectivity maximizers or local maximizers under certain conditions.

Index Terms—multiagent network, consensus algorithm, convergence rate, Laplacian matrix, algebraic connectivity

I. INTRODUCTION

HOW to reach a consensus is a fundamental problem in multiagent networks. Many applications in the real world such as data fusion in sensor networks, flocking, formation flight in space, synchronization of coupled oscillators are closely related to the consensus problem in multiagent networks [2]–[10]. In recent years, various protocols for reaching a consensus have been proposed [2], [3], [11]–[18]. As a representative example, let us consider the consensus protocol proposed by Olfati-Saber and Murray [2]. This protocol is described by a set of linear differential equations, and some fundamental results on the convergence property are obtained [2], [3]. First, the state value of each agent always converges to the average of the initial state values of all agents, if the network satisfies some mild conditions. Second, the rate of convergence is determined by the second smallest eigenvalue of the Laplacian matrix, also known as the algebraic connectivity [19], [20], of the graph, representing the interactions between agents. In general, the algebraic connectivity varies in a wide range depending on the structure of the graph even though the number of vertices and edges is fixed

Part of this work was published in a conference paper [1]. However, the present paper is not just an extended version of it because there are many differences. For example, new theoretical results such as Theorem 4, Theorem 7 and some lemmas needed to prove these theorems are given. Also, background and related works are explained in more detail.

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(see, e.g., [3]). Therefore, finding graphs that maximize the algebraic connectivity under some constraints on the number of vertices, the number of edges, the degree distribution, and so on is a fundamental problem for consensus in multiagent networks.

Kim and Mesbahi [21] considered the problem of maximizing the algebraic connectivity of graphs with weighted edges. In their problem setting, each vertex corresponds to a point mass and the weight of each edge is determined by a function of the distance between two end vertices of the edge. Furthermore, the distance between any two vertices is assumed to be greater than or equal to a positive constant. For this constrained optimization problem, they proposed an iterative algorithm based on a semidefinite program solver, and showed through extensive simulations that it often leads to an optimal solution. Kim [22] considered the problem of finding an edge to be added to a given graph such that the algebraic connectivity of the resulting graph is maximized, and proposed a computationally efficient algorithm for solving the problem. Rafiee and Bayen [23] considered two types of problems related to the optimal topology design for multiagent networks. One is the problem of finding a graph with weighted edges that maximizes the algebraic connectivity under the constraint that the number of edges with nonzero weight is less than or equal to a given positive integer. The other is the problem of finding a graph with weighted edges that minimizes the number of edges with nonzero weight under the constraint that the algebraic connectivity is greater than or equal to a given positive number. They formulated these problems as mixed-integer semidefinite programs (MISDP) and showed that these programs can be solved by using an SDP solver. Dai and Mesbahi [24] considered the optimal topology design problem for dynamic networks in three different scenarios: i) finding a graph with unweighted edges that maximizes the algebraic connectivity under the constraint that the number of edges is bounded from above, ii) finding a graph with weighted edges that maximizes the algebraic connectivity under the assumption that the weight of each edge is determined by a function of two parameters associated with two end vertices and the parameters are bounded from above and below, iii) minimizing the time period required for agents to move from the given initial state to the given final state while obeying the consensus dynamics. All of these problems are formulated as mathematical programming problems and can be solved by appropriate solvers.

The problem of finding optimal graphs has also been studied for the discrete-time consensus protocol. Xiao and Boyd [12] considered the problem of optimizing the communication

weights in the distributed averaging protocol for the fastest convergence, and showed that it can be cast as an SDP. They also introduced an ℓ_1 regularized variant of the problem, and showed that the weights and the topology of communication are optimized simultaneously by solving it. On the other hand, Delvenne [17] *et al.* proved that the optimal topology of communication under the constraint that the out-degree of any vertex is upper bounded by a given positive integer ν is given by a de Bruijn graph. However, this result holds true only for the case where the number of vertices is a power of ν .

The importance of the algebraic connectivity is not restricted to consensus in multiagent networks. The algebraic connectivity is, as its name suggests, a measure that represents how well connected the network is, and closely related to the traditional vertex and edge connectivities [19], [25]. It is thus very useful for designing robust networks. In fact, the algebraic connectivity played an important role in the design of various networks such as computer networks [26]–[29] and air transportation networks [30], [31]. The algebraic connectivity is also related to the performance of some dynamic processes other than consensus on networks [32]–[34]. For example, a network of dynamical units has a more robust synchronized state if the algebraic connectivity is large, and random walks move and disseminate efficiently in networks with large algebraic connectivity [33].

In this paper, we consider the problem of finding undirected and unweighted graphs that maximize or locally maximize the algebraic connectivity in the space of graphs with a fixed number of vertices and edges. We say that a graph locally maximizes the algebraic connectivity if its algebraic connectivity is not less than any graph obtained from it by rewiring only one edge. From a viewpoint of multiagent networks, this can be viewed as the problem of finding topologies of communication between agents with which the network reaches a consensus in the shortest or nearly shortest time, under the constraint that the total power of communication is fixed. This is very similar to the first scenario of Dai and Mesbahi. However, we do not take a mathematical programming approach, but study this problem analytically. We first prove that some well-known classes of graphs such as star graphs, cycle graphs, complete bipartite graphs maximize the algebraic connectivity under certain conditions. We then prove that cycle graphs, complete bipartite graphs, and circulant graphs locally maximize the algebraic connectivity.

The analysis and design of the network topology based on the algebraic connectivity have also been extensively studied in applied mathematics [35]–[47]. We mention here some of the existing results related to the subject of this paper. Fallat and Kirkland [36] studied the algebraic connectivity of unicycle graphs with a given girth, and proved that any graph obtained by adding one edge to a star graph maximizes the algebraic connectivity among all unicycle graphs with the same number of vertices and girth 3. A generalized version of this result will be given in Section III. Belhaiza *et al.* [37] provided some results on graphs with the maximum algebraic connectivity for a given number of vertices and edges. Their results are valid for the case where the number of edges is sufficiently large, while many of the results in the present paper apply to

the case where the number of edges is small in the sense that it is proportional to the number of vertices. Wang *et al.* [43] proved that the maximum of the algebraic connectivity among all graphs with a given number of vertices and a given diameter is achieved by a chain of cliques. Bıyıkođlu and Leydold [45] proved that if a graph has the minimum algebraic connectivity among all connected graphs with a given number of vertices and edges then it must consist of a chain of cliques. Sydney *et al.* [46] recently proposed an iterative method based on edge rewiring to obtain a graph having high algebraic connectivity. In addition to these works, there are a number of theoretical results concerning the algebraic connectivity (see, e.g., [40] and [44]). The present paper not only extends some known results on graphs with the maximum algebraic connectivity to more general ones but also takes a new direction by introducing the problem of finding graphs that locally maximize the algebraic connectivity, which has not been studied before.

The rest of this paper is organized as follows. In Section II, we review the consensus protocol of Olfati-Saber and Murray [2] and its convergence properties. We also demonstrate that the algebraic connectivity of a graph with a fixed number of vertices and edges varies greatly depending on its structure. In Section III, the notion of algebraic connectivity maximizing graph is first introduced, and then some classes of these graphs are presented. In Section IV, the notion of algebraic connectivity locally maximizing graph is introduced first, and then some theoretical results on these graphs are given. Finally, we conclude the paper in Section V.

II. BACKGROUND AND MOTIVATION

We consider networks of n agents labeled from 1 to n that can interact with each other. The set of agents with which agent i can directly interact is denoted by $\mathcal{N}_i \subseteq \{1, 2, \dots, n\} \setminus \{i\}$. Suppose that each agent i has its own state value $x_i(t)$ where t represents time, and increase or decrease it continuously based on the state values of agents in \mathcal{N}_i . Throughout this paper, we assume that interactions between agents are time-invariant (or static) and symmetric, that is, $j \in \mathcal{N}_i$ if and only if $i \in \mathcal{N}_j$. Under this setting, Olfati-Saber and Murray [2] proposed the state update rule described by

$$\dot{x}_i(t) = \sum_{j \in \mathcal{N}_i} (x_j(t) - x_i(t)), \quad i = 1, 2, \dots, n, \quad (1)$$

and showed that the network reaches an average consensus, that is, $\lim_{t \rightarrow \infty} x_i(t) = \sum_{j=1}^n x_j(0)/n$ for $i = 1, 2, \dots, n$, if it is connected. Here, a multiagent network is said to be connected if for any two different agents i and j there is a sequence of agents $i_1(=i), i_2, \dots, i_l(=j)$ such that $i_{k+1} \in \mathcal{N}_{i_k}$ for $k = 1, 2, \dots, l-1$.

Interactions between agents can be represented by a simple undirected graph $G = (V, E)$ where $V = \{1, 2, \dots, n\}$ is the set of vertices representing n agents, and E is the set of edges representing interactions between agents. In this paper, each edge is expressed by an unordered pair of two different vertices like $\{i, j\}$. A pair $\{i, j\}$ is a member of E if and only if agents i and j can directly interact with each other, that is,

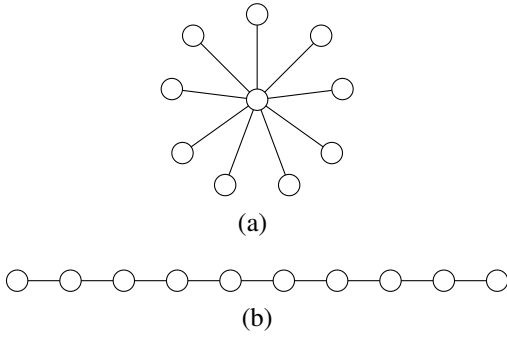


Fig. 1. Two graphs consisting of 10 vertices and nine edges. (a) Star graph. (b) Path graph.

$j \in \mathcal{N}_i$ or equivalently $i \in \mathcal{N}_j$. The Laplacian matrix of a graph $G = (V, E)$ is defined by

$$L = D - A \quad (2)$$

where $A = (a_{ij}) \in \{0, 1\}^{n \times n}$ is the adjacency matrix and $D = \text{diag}(d_1, d_2, \dots, d_n)$ is the degree matrix of the graph G . Note that the components of D are determined by those of A as $d_i = \sum_{j \neq i} a_{ij}$ for $i = 1, 2, \dots, n$. By using the Laplacian matrix L , Eq.(1) can be rewritten as

$$\dot{\mathbf{x}}(t) = -L\mathbf{x}(t) \quad (3)$$

where $\mathbf{x}(t) = (x_1(t), x_2(t), \dots, x_n(t))^T$.

Let the eigenvalues of L be denoted by $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$. It is well known that i) all eigenvalues of L are real and nonnegative, ii) λ_1 is always zero, and iii) λ_2 is nonzero if and only if G is connected (see, e.g., [25]). Therefore, the convergence rate of the consensus algorithm (1) is determined by the second smallest eigenvalue of L , that is, the algebraic connectivity, if the network is connected [3].

The algebraic connectivity of a graph varies greatly depending on its structure, even though the number of vertices and edges is fixed. It is illustrated in [3] that a small-world network with 100 nodes and 300 links reaches an average consensus about 22 times faster than a regular lattice with the same number of nodes and links. Here, we consider a much simpler case where a star graph composed of 10 vertices and a path graph composed of the same number of vertices are compared (see Fig. 1). Note that both graphs have nine edges. The algebraic connectivity of the star graph is 1 and that of the path graph is about 0.0978. This indicates that the former reaches an average consensus about 10 times faster than the latter. In fact, this estimation is confirmed by Fig.2 which illustrates trajectories of state values when $x_i(0) = i$ for $i = 1, 2, \dots, 10$.

In the following sections, by a graph, we mean a simple undirected graph. The set of all graphs composed of n vertices and m edges is denoted by $\mathcal{G}_{n,m}$. For each graph $G \in \mathcal{G}_{n,m}$, the eigenvalues of its Laplacian matrix are denoted by $\lambda_1(G), \lambda_2(G), \dots, \lambda_n(G)$, instead of $\lambda_1, \lambda_2, \dots, \lambda_n$, in order to indicate explicitly the graph under consideration. As mentioned above, we assume without loss of generality that eigenvalues are sorted in ascending order as $0 = \lambda_1(G) \leq$

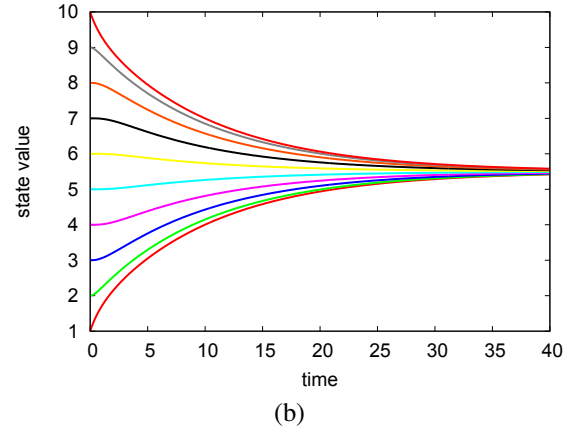
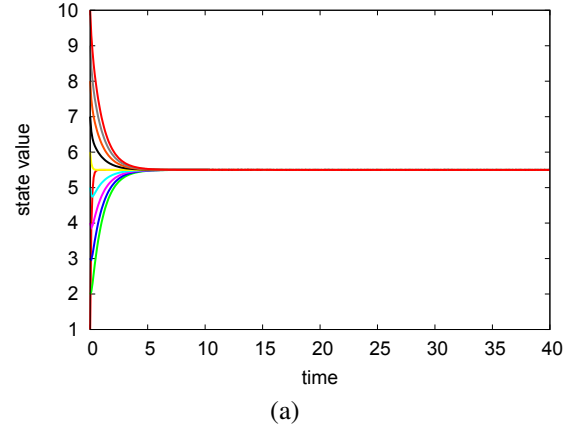


Fig. 2. Trajectories of the state values of 10 agents generated by the consensus algorithm (1) for (a) the star graph and (b) the path graph. Because the algebraic connectivity of the star graph is about 10 times that of the path graph, the convergence rate of the consensus algorithm for the star graph is about 10 times faster than that for the path graph, though both graphs have the same number of vertices and edges.

$\lambda_2(G) \leq \dots \leq \lambda_n(G)$. Then the algebraic connectivity of a graph G is denoted by $\lambda_2(G)$.

III. ALGEBRAIC CONNECTIVITY MAXIMIZING GRAPHS

A. Problem Setting and Brute-Force Search

In this section, we consider the problem of finding a graph $G \in \mathcal{G}_{n,m}$ for each pair (n, m) , such that $\lambda_2(G)$ is not less than any other graphs in $\mathcal{G}_{n,m}$. We call such a graph an Algebraic Connectivity Maximizing (ACM) graph in $\mathcal{G}_{n,m}$. Because the convergence rate of the consensus algorithm (1) is determined by the algebraic connectivity of the graph representing interactions between agents, each ACM graph shows how interactions should be performed by agents for the fastest consensus.

We first identify all ACM graphs in $\mathcal{G}_{n,m}$ for small n by a brute-force search. To be more specific, for each pair (n, m) such that $5 \leq n \leq 7$ and $n - 1 \leq m \leq n(n - 1)/2$, we enumerate all graphs in $\mathcal{G}_{n,m}$, calculate their algebraic connectivity, and determine the graphs that have the highest algebraic connectivity. Results for $n = 5, 6$ and 7 are summarized in Figs. 3, 4 and 5, respectively. From these results, some important facts are observed: i) $\mathcal{G}_{n,m}$ can have

more than one ACM graph, ii) the star graph S_n is the unique ACM graph in $\mathcal{G}_{n,n-1}$ for $n = 5, 6$ and 7 , iii) the cycle graph C_n is an ACM graph in $\mathcal{G}_{n,n}$ for $n = 5$ and 6 , but not for $n = 7$, iv) for $(n, m) = (5, 4), (5, 6), (6, 5), (6, 8), (7, 6), (7, 10)$ and $(7, 12)$, $\mathcal{G}_{n,m}$ has the unique ACM graph which is a complete bipartite graph, v) for $(n, m) = (5, 5), (6, 6), (6, 9), (6, 12)$ and $(7, 14)$, a circulant graph is contained in the set of ACM graphs in $\mathcal{G}_{n,m}$.

The MISDP method [24] can also be used to find an ACM graph in $\mathcal{G}_{n,m}$ for each pair of n and m . However, we should note that it cannot be directly applied to the problem of finding all ACM graphs in $\mathcal{G}_{n,m}$. In fact, the MISDP method generates only one graph, which is isomorphic to the right one shown in Fig. 5(k), when it is applied to the case where $(n, m) = (7, 16)$ [24].

B. Theoretical Analysis

Before proceeding to the main results of this section, we introduce some known results [19], [20], [48] about eigenvalues of the Laplacian matrix. Proofs of Lemmas 1, 2, 3 and 4 can be found in [20]. Proofs of Lemmas 5 and 6 can be found in [19] and [48], respectively.

Lemma 1: Eigenvalues of the Laplacian matrix of the star graph S_n with $n \geq 3$ are given by

$$\lambda_i(S_n) = \begin{cases} 0, & i = 1, \\ 1, & i = 2, 3, \dots, n-1, \\ n, & i = n. \end{cases}$$

Lemma 2: Eigenvalues of the Laplacian matrix of the path graph P_n are given by

$$\lambda_i(P_n) = 2 \left(1 - \cos \frac{\pi(i-1)}{n} \right), \quad i = 1, 2, \dots, n.$$

Lemma 3: Eigenvalues of the Laplacian matrix of the cycle graph C_n are given by

$$\lambda_i(C_n) = 2 \left(1 - \cos \frac{2\pi(\lceil (i-1)/2 \rceil)}{n} \right), \quad i = 1, 2, \dots, n$$

where $\lceil r \rceil$ represents the smallest integer not less than r .

Lemma 4: Eigenvalues of the Laplacian matrix of the complete bipartite graph $K_{k,n-k}$ with $1 \leq k \leq \lfloor n/2 \rfloor$ are given by

$$\lambda_i(K_{k,n-k}) = \begin{cases} 0, & i = 1, \\ k, & i = 2, 3, \dots, n-k, \\ n-k, & i = n-k+1, \dots, n-1, \\ n, & i = n, \end{cases}$$

where $\lfloor r \rfloor$ represents the largest integer not greater than r .

Lemma 5: If G is not a complete graph then $\lambda_2(G) \leq \delta(G)$ where $\delta(G)$ is the minimum degree of G .

Lemma 6: Let $G' \in \mathcal{G}_{n,m+1}$ be a graph obtained by adding an edge to $G \in \mathcal{G}_{n,m}$. Then we have

$$\lambda_1(G) \leq \lambda_1(G') \leq \lambda_2(G) \leq \lambda_2(G') \leq \dots \leq \lambda_n(G) \leq \lambda_n(G').$$

Let us first consider the case where $m = n$. In this case, we obtain the following two results.

Theorem 1: For $n = 3, 4, 5$ and 6 , the cycle graph C_n is an ACM graph in $\mathcal{G}_{n,n}$.

Proof: Let G be any graph in $\mathcal{G}_{n,n}$ with $n \geq 3$. Then, depending on the degree distribution, there are two possible cases: one is that all vertices have degree 2 and the second is that at least one vertex has degree less than 2. In the former case, G must be the cycle graph C_n . The algebraic connectivity of C_n is given by Lemma 3 as

$$\lambda_2(C_n) = 2 \left(1 - \cos \frac{2\pi}{n} \right) \quad (4)$$

which is greater than 1 if $3 \leq n \leq 6$. In the latter case, we have from Lemma 5 that $\lambda_2(G) \leq \delta(G) \leq 1$. Therefore, the cycle graph C_n is an ACM graph in $\mathcal{G}_{n,n}$ if $3 \leq n \leq 6$. ■

Theorem 2: For any $n \geq 6$, any graph obtained by adding an edge to the star graph S_n is an ACM graph in $\mathcal{G}_{n,n}$.

Proof: By Lemmas 1 and 6, the algebraic connectivity of any graph obtained by adding an edge to the star graph S_n with $n \geq 6$ is 1. So it suffices for us to show that $\lambda_2(G) \leq 1$ for any $G \in \mathcal{G}_{n,n}$ with $n \geq 6$. Since the average degree of G is 2, there are two possible cases: one is that all vertices have degree 2 and the other is that at least one vertex has degree less than 2. In the former case, G must be the cycle graph C_n . By this fact and Eq.(4), $\lambda_2(G)$ is not greater than 1 for all $n \geq 6$. In the latter case, by Lemma 5, we have $\lambda_2(G) \leq \delta(G) \leq 1$. Therefore, any graph obtained by adding an edge to the star graph S_n is an ACM graph in $\mathcal{G}_{n,n}$ if $n \geq 6$. ■

Graphs covered by Theorems 1 and 2 are shown in Figs. 3(b), 4(b) and 5(b).

As far as the case where $n \geq 6$ is considered, Theorem 2 is a generalization of a theorem proved by Fallat and Kirkland [36, Theorem 4.14] which says that any graph obtained by adding an edge to the star graph S_n maximizes the algebraic connectivity among all unicycle graphs with n vertices and girth 3.

We next consider complete bipartite graphs.

Theorem 3: For any $n \geq 3$ and any integer k satisfying $1 \leq k \leq \lfloor n/2 \rfloor$ and

$$k - \frac{2k^2}{n} < 1, \quad (5)$$

the complete bipartite graph $K_{k,n-k}$ is an ACM graph in $\mathcal{G}_{n,k(n-k)}$.

Proof: Let G be any graph in $\mathcal{G}_{n,k(n-k)}$. The average degree of G is given by

$$\frac{2m}{n} = \frac{2k(n-k)}{n} = 2k \left(1 - \frac{k}{n} \right) = k + k - \frac{2k^2}{n}.$$

If (5) is satisfied, this quantity is less than $k+1$ and hence the minimum degree of G is at most k . By this fact and Lemma 5, we have $\lambda_2(G) \leq \delta(G) \leq k$. On the other hand, we have from Lemma 4 that $\lambda_2(K_{k,n-k}) = k$. Therefore, the complete bipartite graph $K_{k,n-k}$ is an ACM graph in $\mathcal{G}_{n,k(n-k)}$. ■

Solving the inequality (5) for k under the assumptions that $n \geq 3$ and $1 \leq k \leq \lfloor n/2 \rfloor$, we obtain the following result.

Corollary 1: If two integers n and k satisfy one of the following three conditions then the complete bipartite graph $K_{k,n-k}$ is an ACM graph in $\mathcal{G}_{n,k(n-k)}$.

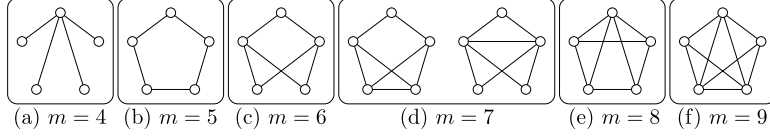


Fig. 3. Algebraic connectivity maximizing graphs in $\mathcal{G}_{5,m}$ ($m = 4, 5, \dots, 9$). The values of algebraic connectivity are (a) 1, (b) 1.382, (c) 2, (d) 2, (e) 3 and (f) 3.

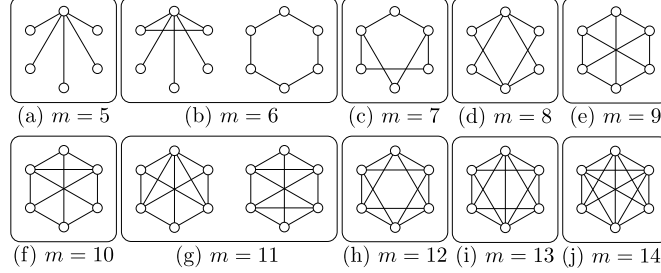


Fig. 4. Algebraic connectivity maximizing graphs in $\mathcal{G}_{6,m}$ ($m = 5, 6, \dots, 14$). The values of algebraic connectivity are (a) 1, (b) 1, (c) 1.268, (d) 2, (e) 3, (f) 3, (g) 3, (h) 4, (i) 4, (j) 4.

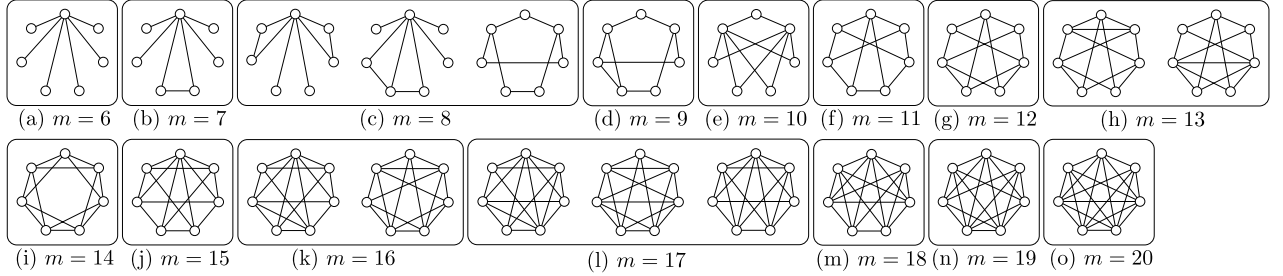


Fig. 5. Algebraic connectivity maximizing graphs in $\mathcal{G}_{7,m}$ ($m = 6, 7, \dots, 20$). The values of algebraic connectivity are (a) 1, (b) 1, (c) 1, (d) 1.586, (e) 2, (f) 2.139, (g) 3, (h) 3, (i) 3.198, (j) 4, (k) 4, (l) 4, (m) 5, (n) 5, (o) 5.

- 1) $3 \leq n \leq 7$ and $1 \leq k \leq \lfloor n/2 \rfloor$
- 2) $n \geq 8$ and $k = 1$
- 3) $n \geq 8$ and $(n + \sqrt{n^2 - 8n})/4 < k \leq \lfloor n/2 \rfloor$

Because the inequality (5) is satisfied with $k = 1$ for any $n \geq 3$, we have the following result as a special case of Theorem 3.

Corollary 2 (Mohar [48]): For any $n \geq 3$, the star graph S_n is an ACM graph in $\mathcal{G}_{n,n-1}$.

Graphs covered by Theorem 3 are shown in Figs. 3(a), 3(c), 4(a), 4(d), 4(e), 5(a), 5(e), and 5(g). These graphs correspond to the cases where $(n, k) = (5, 1), (5, 2), (6, 1), (6, 2), (6, 3), (7, 1), (7, 2)$ and $(7, 3)$. The inequality (5) is also satisfied with $(n, k) = (8, 3), (8, 4), (9, 4), (10, 4), (10, 5)$, and so on. Looking at these pairs, we expect that the complete bipartite graph $K_{k,n-k}$ is an ACM graph in $\mathcal{G}_{n,k(n-k)}$ if k is close to $n/2$. In fact, we obtain the following result as a special case of Theorem 3.

Corollary 3: For any $n \geq 3$ and any integer k satisfying

$$\left\lfloor \frac{n-1}{2} \right\rfloor \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor, \quad (6)$$

the complete bipartite graph $K_{k,n-k}$ is an ACM graph in $\mathcal{G}_{n,k(n-k)}$.

Proof: Suppose first that n is an even number. Then n can be expressed as $n = 2l$ where l is an integer not less than 2. Furthermore, the condition (6) is equivalent to $l-1 \leq k \leq l$. If $n = 2l$ and $k = l-1$, we have

$$\begin{aligned} k - \frac{2k^2}{n} &= l-1 - \frac{2(l-1)^2}{2l} \\ &= \frac{l-1}{l} \{l - (l-1)\} \\ &= \frac{l-1}{l} < 1. \end{aligned}$$

If $n = 2l$ and $k = l$, we have

$$k - \frac{2k^2}{n} = l - \frac{2l^2}{2l} = l - l = 0 < 1.$$

Namely, if n is an even number and k is either $n/2 - 1$ or $n/2$, the inequality (5) is satisfied. Suppose next that n is an odd number. Then n can be expressed as $n = 2l - 1$ where l is any integer not less than 2. Furthermore, the condition (6) is equivalent to $k = l - 1$. If $n = 2l - 1$ and $k = l - 1$, the inequality (5) is satisfied because

$$\begin{aligned} k - \frac{2k^2}{n} &= l-1 - \frac{2(l-1)^2}{2l-1} \\ &= \frac{l-1}{2l-1} \{(2l-1) - 2(l-1)\} \end{aligned}$$

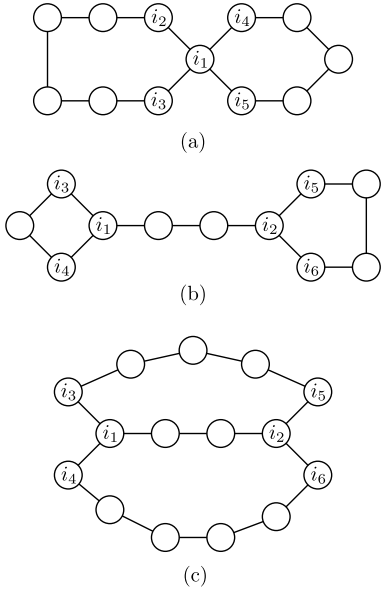


Fig. 6. Three possible topologies of a connected graph $G \in \mathcal{G}_{n,n+1}$ with $\delta(G) \geq 2$.

$$= \frac{l-1}{2l-1} < 1.$$

Therefore, by Theorem 3, any complete bipartite graph $K_{k,n-k}$ satisfying (6) is an ACM graph in $\mathcal{G}_{n,k(n-k)}$. ■

We next consider graphs obtained by adding a small number of edges to a star graph. The following two lemmas are needed in later discussion.

Lemma 7: Let n and k be positive integers such that $n \geq 4$ and $1 \leq k \leq n$. If $G \in \mathcal{G}_{n,n+k}$ is connected and $\delta(G) \geq 2$ then there are $2k$ edges in G such that the removal of these edges results in a linear forest with k components, where a linear forest is a graph in which each component is a path.

Proof: We prove Lemma 7 by mathematical induction. As the initial step, we will prove the following two statements: i) if $G \in \mathcal{G}_{4,4+k}$ and $k \in \{1, 2\}$ then there are $2k$ edges in G such that the removal of these edges results in a linear forest with k components, ii) for any $n \geq 5$, if $G \in \mathcal{G}_{n,n+1}$ is connected and $\delta(G) \geq 2$ then there exist two edges in G such that the removal of these edges results in a path graph. For the first statement, by considering any graph in $\mathcal{G}_{4,5}$ and the unique graph in $\mathcal{G}_{4,6}$, one can immediately see that it is true. So we consider only the second one. Because G has $n+1$ edges and $\delta(G) \geq 2$, the distribution of degrees is restricted to two cases. The first case is that one vertex has degree four and other $n-1$ vertices have degree two. In this case, G always has the topology shown in Fig. 6 (a). It is easily seen from the figure that we can obtain a path graph by removing two edges, say, $\{i_1, i_2\}$ and $\{i_1, i_4\}$. The second case is that two vertices have degree three and other $n-2$ vertices have degree two. In this case, G always has one of two possible topologies shown in Fig. 6 (b) and (c). In either type of graph, we can obtain a path graph by removing two edges, say, $\{i_1, i_3\}$ and $\{i_2, i_6\}$.

As the inductive step, we will prove that the statement of Lemma 7 holds true for $n = n^*$ and $k = k^*$ under the assumption that it holds true for $n = 4, 5, \dots, n^* - 1$ and

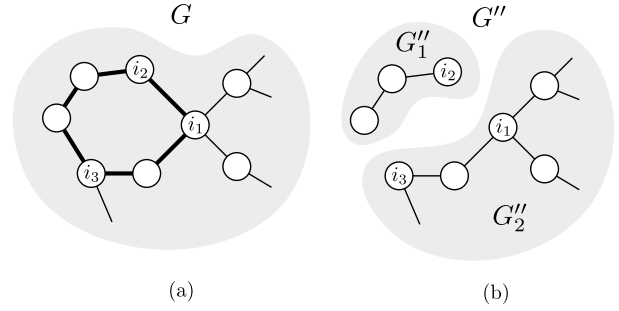


Fig. 7. Relationship between G and G'' . (a) G has a cycle containing a vertex with degree greater than two. (b) G'' has two connected components G''_1 and G''_2 .

for $n = n^*$ and $k = 1, 2, \dots, k^* - 1$, where n^* is any integer greater than four and k^* is any integer such that $2 \leq k^* \leq n^*$. Because G is not a tree, it has at least one cycle. Moreover, because G itself is not a cycle graph, any cycle in G contains at least one vertex with degree greater than two. Hence G has an edge $\{i_1, i_2\}$ such that it is in a cycle and the vertex i_1 has degree greater than two, as shown in Fig. 7 (a). Let G' be the graph obtained from G by removing the edge $\{i_1, i_2\}$. Then, G' belongs to $\mathcal{G}_{n^*, n^* + k^* - 1}$, G' is connected, and all vertices except i_2 have degree greater than one in G' . In the remainder of the proof, we shall consider two possible cases depending on the degree of the vertex i_2 .

First, let us consider the case where i_2 has degree greater than one in G' . By the assumption, there are $2(k^* - 1)$ edges in G' such that a linear forest with $k^* - 1$ components is obtained from G' by removing those edges. Note here that at least one component has more than one vertex. This is because $k^* - 1$ is less than n^* . So, by removing an edge in such a component, we can finally obtain a linear forest with k^* components. The total number of edges removed from G is $1 + 2(k^* - 1) + 1 = 2k^*$.

Next, let us consider the case where i_2 has degree one in G' . In this case, there exists a path in G' starting at i_2 , visiting some vertices with degree two, and ends at a vertex i_3 with degree greater than two. Let n_1 be the length of the path. Then we have $1 \leq n_1 \leq n^* - 1$. Let G'' be the graph obtained from G' by removing the last edge of the path, that is, the edge having i_3 as one of its endpoints. Then G'' has two components; one, denoted by G''_1 , is a path having n_1 vertices and the other, denoted by G''_2 , is a connected graph having $n^* - n_1$ vertices and $n^* - n_1 + k^* - 1$ edges, as shown in Fig. 7 (b). If $k^* - 1 \leq n^* - n_1$ then, by the assumption, there are $2(k^* - 1)$ edges in G''_2 such that the removal of these edges from G''_2 results in a linear forest with $k^* - 1$ components. This implies that we can obtain a linear forest with k^* components from G by removing $2 + 2(k^* - 1) = 2k^*$ edges. If $k^* - 1 > n^* - n_1$ then we obtain a linear forest with $n^* - n_1$ components from G''_2 by removing all edges. Also, we obtain a linear forest with $k^* - n^* + n_1$ components from G''_1 by removing $k^* - n^* + n_1 - 1$ edges. Here, we should note that $k^* - n^* + n_1 - 1$ is positive due to the assumption that $k^* - 1 > n^* - n_1$ and less than n_1 due to the assumption that $k^* \leq n^*$. In summary, we can obtain a linear forest with $(n^* - n_1) + (k^* - n^* + n_1) = k^*$ components from G by

removing $2 + (n^* - n_1 + k^* - 1) + (k^* - n^* + n_1 - 1) = 2k^*$ edges. ■

Lemma 8: Let n and k be positive integers such that $n \geq 5k + 4$. If $G \in \mathcal{G}_{n,n-k}$ is a linear forest with k components then $\lambda_{2k+2}(G) \leq 1$.

Proof: Let G_j be the j -th component of G and let $\lambda_{j1}(G_j) \leq \lambda_{j2}(G_j) \leq \dots \leq \lambda_{jn_j}(G_j)$ be eigenvalues of the Laplacian matrix of G_j , where $n_j = |V(G_j)|$. Then, $\{\lambda_i(G) \mid i = 1, 2, \dots, n\} = \{\lambda_{ji}(G_j) \mid j = 1, 2, \dots, k; i = 1, 2, \dots, n_j\}$. It follows from Lemma 2 that

$$\lambda_{ji}(G_j) = 2 \left(1 - \cos \frac{\pi(i-1)}{n_j} \right), \quad i = 1, 2, \dots, n_j.$$

Because $\lambda_{ji}(G_j) \leq 1$ if and only if $i \leq \lfloor \frac{n_j}{3} \rfloor + 1$, the number of $i \in \{1, 2, \dots, n\}$ such that $\lambda_i(G) \leq 1$ is given by

$$\sum_{j=1}^k \left(\left\lfloor \frac{n_j}{3} \right\rfloor + 1 \right) = \sum_{j=1}^k \left\lfloor \frac{n_j}{3} \right\rfloor + k.$$

Here, because $n = \sum_{j=1}^k n_j = \sum_{j=1}^k (3 \lfloor \frac{n_j}{3} \rfloor + (n_j \bmod 3)) \geq 5k + 4$ and $n_j \bmod 3 \leq 2$, we have

$$\sum_{j=1}^k \left\lfloor \frac{n_j}{3} \right\rfloor + k \geq 2k + 2.$$

Therefore, we conclude that $\lambda_{2k+2}(G) \leq 1$. ■

We are now ready to present one of the main results of this section.

Theorem 4: Let n and k be any positive integers such that $n \geq 5k + 4$. The graph obtained by adding $k + 1$ edges to the star graph S_n is an ACM graph in $\mathcal{G}_{n,n+k}$.

Proof: We first note that n is necessarily greater than or equal to 9. Let G^* be any graph obtained from the star graph S_n by adding $k + 1$ edges. Here, we see that $k + 1 < n - 3$ because the inequalities $k + 1 \leq (n - 4)/5 + 1 = (n + 1)/5 < n - 3$ follows from $n \geq 5k + 4$ and $n \geq 9$. From this fact together with Lemmas 1 and 6, we have $\lambda_2(G^*) = 1$. Hence, it suffices for us to show that $\lambda_2(G) \leq 1$ for any $G \in \mathcal{G}_{n,n+k}$. Suppose first that $\delta(G) = 1$. Then, by Lemma 5, we have $\lambda_2(G) \leq \delta(G) = 1$. Suppose next that $\delta(G) \geq 2$. Then, by Lemma 7, we can choose $2k$ edges from $E(G)$ such that a linear forest G' composed of k components is obtained from G by removing the $2k$ edges. Furthermore, by Lemmas 6 and 8, we have $\lambda_2(G) \leq \lambda_{2k+2}(G') \leq 1$. ■

Theorem 4 can be rewritten in terms of the number of vertices and edges as follows.

Corollary 4: Let n and m be any positive integers such that $n \geq 9$ and $n + 1 \leq m \leq (6n - 4)/5$. The graph obtained by adding $m - (n - 1)$ edges to the star graph S_n is an ACM graph in $\mathcal{G}_{n,m}$.

IV. ALGEBRAIC CONNECTIVITY LOCALLY MAXIMIZING GRAPHS

A. Problem Setting

In the previous section, we have given some classes of ACM graphs. However, Figs. 3–5 contain many graphs that are not included in any of these classes. In this sense, the results in

the previous section are not sufficient to characterize all ACM graphs.

In this section, we consider the problem of finding graphs $G \in \mathcal{G}_{n,m}$ such that the algebraic connectivity of G is not less than that of any graph obtained from G by rewiring an edge, that is, removing an edge from and adding a new edge to G . In the remainder of this section, such a graph is called an Algebraic Connectivity Locally Maximizing (ACLM) graph in $\mathcal{G}_{n,m}$. Also, the set of all graphs obtained from $G \in \mathcal{G}_{n,m}$ by rewiring an edge is called the neighborhood of G in $\mathcal{G}_{n,m}$, and denoted by $\mathcal{N}_{n,m}(G)$. Then a graph $G \in \mathcal{G}_{n,m}$ is an ACLM graph if and only if $\lambda_2(G) \geq \lambda_2(G')$ for all $G' \in \mathcal{N}_{n,m}(G)$. Hence, in order to prove that G is an ACLM graph in $\mathcal{G}_{n,m}$, we do not have to consider all graphs in $\mathcal{G}_{n,m}$ but just in $\mathcal{N}_{n,m}(G)$ which is much smaller than $\mathcal{G}_{n,m}$.

It is apparent from the definitions of ACM and ACLM graphs that if G is an ACM graph in $\mathcal{G}_{n,m}$ then it is necessarily an ACLM graph in $\mathcal{G}_{n,m}$. In other words, the set of all ACM graphs in $\mathcal{G}_{n,m}$ is contained in the set of all ACLM graphs in $\mathcal{G}_{n,m}$. Although we do not directly consider ACM graphs in this section, we may obtain new insight into ACM graphs through the analysis of ACLM graphs. In particular, for some graphs in Figs. 3–5 that the results in the previous section cannot be applied to, we may be able to prove that they are ACLM graphs.

Another important aspect of ACLM graphs is that each of them has a locally optimal topology in the sense that the algebraic connectivity cannot be increased no matter which edge is rewired. This means that any local search algorithm based on edge rewiring (see, e.g., [46]) stops at an ACLM graph. Each ACLM graph is thus expected to have a relatively high algebraic connectivity. Therefore, the analysis of ACLM graphs is important not only for understanding ACM graphs but also for identifying suboptimal topologies.

B. Theoretical Analysis

We first consider cycle graphs.

Theorem 5: The cycle graph C_n is an ACLM graph in $\mathcal{G}_{n,n}$.

Proof: Any graph obtained from C_n by removing an edge is isomorphic to the path graph P_n . Let G be any graph belonging to $\mathcal{N}_{n,n}(C_n)$. Then G is isomorphic to a graph obtained from P_n by adding an edge. By Lemma 6, we have

$$\lambda_2(P_n) \leq \lambda_2(G) \leq \lambda_3(P_n). \quad (7)$$

Also, by Lemmas 2 and 3, we have

$$\lambda_3(P_n) = 2 \left(1 - \cos \frac{2\pi}{n} \right) = \lambda_2(C_n). \quad (8)$$

Therefore, we have from (7) and (8) that $\lambda_2(G) \leq \lambda_2(C_n)$. Since G is any graph in $\mathcal{N}_{n,n}(C_n)$, we conclude that the cycle graph C_n is an ACLM graph in $\mathcal{G}_{n,n}$. ■

Because the cycle graph C_n is not an ACM graph for $n \geq 7$, this result implies that not all ACLM graphs are ACM graphs.

We next consider complete bipartite graphs.

Theorem 6: For any positive integers n and k satisfying $2 \leq k \leq \lfloor n/2 \rfloor$, the complete bipartite graph $K_{k,n-k}$ is an ACLM graph in $\mathcal{G}_{n,k(n-k)}$.

Proof: Let G be any graph in $\mathcal{N}_{n,k(n-k)}(K_{k,n-k})$ with $2 \leq k \leq \lfloor n/2 \rfloor$. It is easily seen that G has at least one vertex of which degree is k . By this fact and Lemma 5, we have $\lambda_2(G) \leq \delta(G) \leq k$. On the other hand, we have from Lemma 4 that $\lambda_2(K_{k,n-k}) = k$. Therefore, the complete bipartite graph $K_{k,n-k}$ is an ACLM graph in $\mathcal{G}_{n,k(n-k)}$. ■

Theorems 3 and 6 are closely related to a conjecture made by Kolokolnikov [47, Conjecture 1.5] which asserts that the complete bipartite graph $K_{2,n-2}$ is an ACM graph. In fact, Theorem 3 proves that it holds true for $n \leq 7$ and Theorem 6 indicates that it may also be true for $n \geq 8$. However, the conjecture still remains an open problem.

We finally consider circulant graphs. A circulant graph is a graph such that its adjacency matrix is a circulant matrix.

Theorem 7: If $G \in \mathcal{G}_{n,m}$ is a circulant graph with n being odd then G is an ACLM graph in $\mathcal{G}_{n,m}$. If $G \in \mathcal{G}_{n,m}$ is a circulant graph with n being even and if the adjacency matrix $A = (a_{ij})$ of G satisfies

$$2 \sum_{j=2}^{\frac{n}{2}} a_{1j} (-1)^j + a_{1, \frac{n}{2}+1} (-1)^{\frac{n}{2}} > 0 \quad (9)$$

then G is an ACLM graph in $\mathcal{G}_{n,m}$.

Proof: If the Laplacian matrix of G is a circulant matrix, its eigenvalues can be expressed in an explicit form as

$$\psi_i = d_1 - \sum_{j=2}^n a_{1j} \cos\left(\frac{2\pi i(j-1)}{n}\right), \quad i = 0, 1, \dots, n-1$$

where $d_1 = \sum_{j=2}^n a_{1j}$. Here, we should note that $\psi_0 = 0$ and $\psi_i = \psi_{n-i}$ for $i = 1, 2, \dots, n-1$. We should also note that $a_{1j} = a_{1, n+2-j}$ for $j = 2, 3, \dots, \lfloor \frac{n+1}{2} \rfloor$ because A is not only circulant but also symmetric. Let us first consider the case where n is odd. In this case, we have $\lambda_2(G) = \lambda_3(G), \lambda_4(G) = \lambda_5(G), \dots, \lambda_{n-1}(G) = \lambda_n(G)$. Let $G' \in \mathcal{G}_{n,m+1}$ be any graph obtained from G by adding an edge, and $G'' \in \mathcal{G}_{n,m}$ be any graph obtained from G' by removing an edge. Then we have from Lemma 6 that $\lambda_2(G) \leq \lambda_2(G') \leq \lambda_3(G)$ and $\lambda_1(G') \leq \lambda_2(G'') \leq \lambda_2(G')$. These inequalities and the fact that $\lambda_2(G) = \lambda_3(G)$ leads to $\lambda_2(G'') \leq \lambda_2(G)$, which means that G is an ACLM graph in $\mathcal{G}_{n,m}$. Let us next consider the case where n is even. In this case, we have $\psi_i = \psi_{n-i}$ for $i = 1, 2, \dots, \frac{n}{2} - 1$ and

$$\begin{aligned} \psi_{\frac{n}{2}} &= d_1 - \sum_{j=2}^n a_{1j} \cos \pi(j-1) \\ &= d_1 + 2 \sum_{j=2}^{\frac{n}{2}} a_{1j} (-1)^j + a_{1, \frac{n}{2}+1} (-1)^{\frac{n}{2}}. \end{aligned}$$

If G satisfies (9) then we have from Lemma 5 that $\lambda_2(G) \leq \delta(G) = d_1 < \psi_{\frac{n}{2}}$. Hence there exists an $i \in \{1, 2, \dots, \frac{n}{2} - 1\}$ such that $\psi_i = \psi_{n-i} = \min_{1 \leq j \leq n-1} \{\psi_j\}$, which implies that $\lambda_2(G) = \lambda_3(G)$. Then, by using the same argument as above, we can conclude that G is an ACLM graph in $\mathcal{G}_{n,m}$. ■

As a special case of Theorem 7, we have the following lemma.

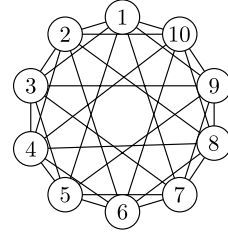


Fig. 8. A graph $G \in \mathcal{G}_{10,30}$ that is a circulant graph but not an ACLM graph.

Corollary 5: If $G \in \mathcal{G}_{n,m}$ is a circulant graph with n being even and the adjacency matrix $A = (a_{ij})$ of G satisfies

$$a_{1j} = \begin{cases} 1, & j = 2, \frac{n}{2} + 1, n, \\ 0, & \text{otherwise,} \end{cases}$$

then G is an ACLM graph in $\mathcal{G}_{n,m}$.

Proof: If G satisfies the conditions then the left-hand side of (9) becomes $2 + (-1)^{\frac{n}{2}}$, which is always positive. ■

Note that not all circulant graphs are ACLM graphs. As a simple example, let us consider the circulant graph $G \in \mathcal{G}_{10,30}$ shown in Fig. 8. Substituting $a_{12} = a_{13} = a_{15} = 1$ and $a_{14} = a_{16} = 0$ into the left-hand side of (9), we have

$$2 \sum_{j=2}^{\frac{n}{2}} a_{1j} (-1)^j + a_{1, \frac{n}{2}+1} (-1)^{\frac{n}{2}} = -2 < 0,$$

which suggests the possibility that G is not an ACLM graph. In fact, we have $\lambda_2(G) (= 4) < \lambda_2(G') (= 4.1391421 \dots)$, where G' is the graph obtained from G by removing the edge $\{1, 3\}$ and adding the edge $\{1, 6\}$.

C. How Close are ACLM graphs to ACM graphs?

For each of the ACLM graphs presented in this section, we see how close its algebraic connectivity is to that of ACM graphs with the same number of vertices and edges.

We first consider the cycle graph C_n . As shown in Figs. 3 and 4, C_n is an ACM graph in $\mathcal{G}_{n,n}$ for $n = 5$ and 6. However, its algebraic connectivity $\lambda_2(C_n) = 2(1 - \cos 2\pi/n)$ converges to 0 as n goes to infinity. On the other hand, the algebraic connectivity of any ACM graph in $\mathcal{G}_{n,n}$ is 1, as shown in the proof of Theorem 2. Therefore, we can conclude that the cycle graph C_n is far from the ACM graph for large n .

We next consider the complete bipartite graph $K_{k,n-k}$ with $2 \leq k \leq \lfloor n/2 \rfloor$. As mentioned in Section III, $K_{k,n-k}$ is an ACM graph in $\mathcal{G}_{n,k(n-k)}$ for $(n, k) = (5, 2), (6, 2), (6, 3), (7, 2), (7, 3), (8, 3), (8, 4), (9, 4), (10, 4)$ and $(10, 5)$. The algebraic connectivity of $K_{k,n-k}$ is k (see Lemma 4). On the other hand, the algebraic connectivity of any $G \in \mathcal{G}_{n,k(n-k)}$ is less than $2k$ because the minimum degree of G is less than or equal to $2k(n-k)/n = 2k - 2k^2/n < 2k$. This means that the algebraic connectivity of $K_{k,n-k}$ with $2 \leq k \leq \lfloor n/2 \rfloor$ is greater than the half of that of ACM graphs in $\mathcal{G}_{n,k(n-k)}$.

We finally consider circulant graphs. As shown in Figs. 3, 4 and 5, all circulant graphs in $\mathcal{G}_{n,m}$ with $5 \leq n \leq 7$ except C_7 are ACM graphs in $\mathcal{G}_{n,m}$. However, it is difficult to draw a general conclusion about circulant graphs because a variety

of graphs are contained. So we focus our attention on two extreme cases. One is the case where $a_{12} = a_{1n} = 1$ and $a_{1j} = 0$ for $j = 3, 4, \dots, n - 1$. The other is the case where n is even, $a_{1j} = 1$ if j is even, and $a_{1j} = 0$ otherwise. The former corresponds to the cycle graph C_n which is far from the ACM graph for large n . The latter corresponds to the complete bipartite graph $K_{n/2, n/2}$ which is an ACM graph in $\mathcal{G}_{n, n^2/4}$ (see Corollary 3).

V. CONCLUDING REMARKS

We have studied the problem of finding undirected and unweighted graphs that maximize the algebraic connectivity in the space of such graphs with a fixed number of vertices and edges. This problem is closely related to consensus and other dynamic processes in networked systems. We have introduced notions of ACM and ACLM graphs, and presented several theoretical results on these graphs. The results are summarized in Tables I and II where some classes of ACM and ACLM graphs are presented. Tables I and II also show the average degree and its minimum and maximum values for each class of graphs. The authors believe that these results provide useful information for those who want to design some network so that it has the highest or a relatively high algebraic connectivity. However, because Tables I and II cover only a small part of the collection of ACM graphs obtained by a brute-force search, it is still needed to identify other classes of ACM or ACLM graphs as many as possible. Also, in view of the application to networks of agents, it is important to consider the problem of maximizing algebraic connectivity under some constraint on the degree of each vertex. For example, it is interesting to focus our attention on regular graphs.

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TABLE I
LIST OF ACM GRAPHS

Class	Condition	$d_{\text{ave}} (= \frac{2m}{n})$	Minimum of d_{ave}	Maximum of d_{ave}
Star graph S_n	none	$2 - \frac{2}{n}$	$2 - \frac{2}{n}$	$2 - \frac{2}{n}$
Cycle graph C_n	$n \leq 6$	2	2	2
Star graph plus one edge	$n \geq 6$	2	2	2
Star graph plus $k + 1$ edges ($k \geq 1$)	$n \geq 5k + 4$	$2 + \frac{k}{n}$	$2 + \frac{1}{n}$	$\frac{11}{5} - \frac{4}{5n}$
Complete bipartite graph $K_{k, n-k}$ ($2 \leq k \leq \lfloor \frac{n}{2} \rfloor$)	$k - \frac{2k^2}{n} < 1$	$\frac{2k(n-k)}{n}$	$\frac{n}{2} - \frac{2}{n}$ (for even $n \geq 10$) $\frac{n}{2} - \frac{1}{2n}$ (for odd $n \geq 11$)	$\frac{n}{2}$ (for even $n \geq 10$) $\frac{n}{2} - \frac{1}{2n}$ (for odd $n \geq 11$)

TABLE II
LIST OF ACLM GRAPHS

Class	Condition	$d_{\text{ave}} (= \frac{2m}{n})$	Minimum of d_{ave}	Maximum of d_{ave}
Cycle graph C_n	none	2	2	2
Complete bipartite graph $K_{k, n-k}$ ($2 \leq k \leq \lfloor \frac{n}{2} \rfloor$)	none	$\frac{2k(n-k)}{n}$	$4 - \frac{8}{n}$	$\frac{n}{2}$ (for even n) $\frac{n}{2} - \frac{1}{2n}$ (for odd n)
Circulant graph with an odd number of vertices	none	d_1	2	$n - 1$
Circulant graph with an even number of vertices	Eq. (9)	d_1	2	$n - 1$ (for $n \bmod 4 = 0$) $n - 2$ (for $n \bmod 4 \neq 0$)

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