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Sufficient Conditions for One-Dimensional Cellular Neural Networks to Perform Connected Component Detection ⋆

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Abstract

It is well known that one-dimensional cellular neural networks (1-D CNNs) with the template $A = [1, 2, -1]$ can perform connected component detection (CCD). However this has been confirmed only by numerical and laboratory experiments. In this paper, sufficient conditions for 1-D CNNs to perform CCD are obtained through theoretical analysis. Main result shows that a wide class of templates including $A = [1, 2, -1]$ can be used for CCD.

Key words: cellular neural network, connected component detection, template, local regularity, convergence
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1 Introduction

Cellular neural networks (CNNs) proposed by Chua and Yang [1] have many applications in the field of image processing [2]. As an important application of one-dimensional (1-D) CNNs, this paper focuses on connected component

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detection (CCD) [3] which is a task to transform a given 1-D black-and-white image into another black-and-white image such that the number of isolated black pixels in the output image is equal to the number of blocks of consecutive black pixels in the input image (a more rigorous definition is given in Section 2). Matsumoto *et al.* [3] first showed via computer simulation that a 1-D CNN with the template $A = [1, 2, -1]$ can perform CCD. This fact was also confirmed experimentally by Cruz and Chua with their CNN chips [4].

Dynamical behavior of 1-D CNNs has been extensively studied so far [5–14] in relation to CCD. Zou and Nossek [6] considered a 1-D CNN with the antisymmetric template $A = [s, p, -s]$ $(p > 1)$ and proved under the zero boundary condition that it is completely stable if $p - 1 > 2|s|$ and that it possesses no stable equilibrium point if $p - 1 < |s|$. Thiran *et al.* [11] introduced the concepts of local diffusion and global propagation into the dynamical behavior of 1-D CNNs and proved that 1-D CNNs with the template $A = [r, p, s]$ $(p > 1)$ has a local diffusion behavior if $p-1 > |s-r|$. They also studied in detail 1-D CNNs working in the local diffusion mode and presented many results on the number of equilibrium points and the complete stability. On the other hand, Setti *et al.* [12] proved that 1-D CNNs with the template $A = [r, p, s]$ $(p > 1)$ has a global propagation behavior if $p - 1 < |s - r|$. They also studied in detail 1-D CNNs working in the global propagation mode and presented many results on the existence of periodic solutions and their stability under the periodic boundary condition. It should be noted that $A = [1, 2, -1]$ satisfies the condition for the global propagation mode. De Sandre [13] proved that 1-D CNNs with the antisymmetric template $A = [s, p, -s]$ $(p > 1)$ are completely stable under the fixed boundary condition if $p - 1 > |s| \times 1.25670414 \cdots$.

Although various properties of 1-D CNNs have been clarified so far as stated above, no one has proved yet the validity of the template $A = [1, 2, -1]$ for CCD. There are two main reasons which make the analysis difficult. One is that a 1-D CNN has to work in the global propagation mode, because it is apparent from the definition of CCD that the color of each pixel in the output image depends not only on those of its neighbors but also all pixels in the input image. The other is that the state transient must not be monotonic [16], because some pixels change their color from black to white and others from white to black in the process of CCD.

In this paper, we consider 1-D CNNs with the opposite-sign template $A = [r, p, -s]$ $(r > 0,\ p > 1,\ s > 0)$ and provide sufficient conditions for such 1-D CNNs to perform CCD under the fixed boundary condition $y_0(t) = y_{n+1}(t) = -1$ where n is the number of cells. As shown in Section 3, the sufficient conditions are satisfied not only with $A = [1, 2, -1]$ but also with a wide class of templates. This is the main contribution of this paper. A key idea used in our approach is to restrict ourselves to those 1-D CNNs which are locally regular. [17,18]. In locally regular 1-D CNNs, $|y_i(t)|$ and $|y_{i+1}(t)|$ never become less than 1 at the same time, which makes the analysis easier. We first derive sufficient conditions for 1-D CNNs to be locally regular and next show that 1-D CNNs can perform CCD under these conditions.

Although some results of this paper were presented at two conferences [19,20], proofs in those conference papers were not rigorous due to the limited space. So this paper is the first to provide a complete proof of the main result. Also, some new material is given for better understanding of the main result and future works.

2 Problem Formulation

2.1 CNN Model

Let us consider simple 1-D CNNs described by the set of differential equations:

$$\dot{x}_i(t) = -x_i(t) + a_{-1}y_{i-1}(t) + a_0 y_i(t) + a_1 y_{i+1}(t), \quad i = 1, 2, \ldots, n \tag{1}$$

where $x_i(t)$ and $y_i(t)$ represent the state and output of the i-th cell at time t, respectively, and the dot means time derivative. The relationship between $x_i(t)$ and $y_i(t)$ is given by

$$y_i(t) = f(x_i(t)) \triangleq \frac{1}{2}(|x_i(t) + 1| - |x_i(t) - 1|)\,. \tag{2}$$

In this paper, we focus our attention on 1-D CNNs (1) having the opposite-sign template:

$$A = [a_{-1}, a_0, a_1] = [r, p, -s] \quad (r > 0,\ p > 1,\ s > 0) \tag{3}$$

under the fixed boundary condition:

$$y_0(t) = y_{n+1}(t) = -1, \quad \forall t \geq 0\,. \tag{4}$$

Two vectors $\boldsymbol{x}(t) = (x_1(t), x_2(t), \ldots, x_n(t)) \in \mathbb{R}^n$ and $\boldsymbol{y}(t) = (y_1(t), y_2(t), \ldots, y_n(t)) \in [-1, 1]^n$ represent the state and output of the 1-D CNN at time t, respectively. Since each component of the output $\boldsymbol{y}(t)$ is bounded by -1 and 1, $\boldsymbol{y}(t)$ can be identified with a 1-D gray-scale image composed of n pixels by regarding 1 and -1 as black and white, respectively. We therefore say that $y_i(t)$ is black, white, and gray if $y_i(t) = 1$, $y_i(t) = -1$, and $|y_i(t)| < 1$, respectively. Throughout this paper, we assume that the initial output $\boldsymbol{y}(0)$ is a black-and-white (or binary) image, that is,

$$|y_i(0)| = 1, \quad i = 1, 2, \ldots, n \tag{5}$$

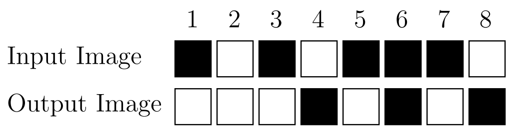

Fig. 1. Example of connected component detection.

which is equivalent to that $|x_i(0)| \geq 1$ for $i = 1, 2, \ldots, n$.

2.2 Connected Component Detection

In order to make our later discussion clear, we give here a rigorous definition of CCD.

Definition 1 *Given a 1-D binary image with n pixels, CCD is to output another binary image with the same size as the original, which satisfies the following properties:*

(1) The number of black pixels is equal to the number of blocks of consecutive black pixels in the given image.
(2) The n-th pixel is black unless all pixes in the given image are white.
(3) All black pixels are isolated.
(4) There exists exactly one white pixels between two neighboring black pixels.

An example of CCD is shown in Fig.1 where three connected components in the input image are represented by three isolated black pixels in the output image.

2.3 Problem

A 1-D CNN is considered to perform an image processing task if we take the initial output $\boldsymbol{y}(0)$ as the input image and the final output $\lim_{t\to\infty} \boldsymbol{y}(t)$ as the output image. Apparently the output $\boldsymbol{y}(t)$ must converge in order for us to get the output image. We now specify the relationship between 1-D CNNs and CCD.

Definition 2 *A 1-D CNN is said to perform CCD if $\boldsymbol{y}(t)$ converges to a binary image which is identical to the result of CCD applied to $\boldsymbol{y}(0)$ for any initial state $\boldsymbol{x}(0)$ such that $|x_i(0)| \geq 1$ for $i = 1, 2, \ldots, n$.*

The problem we tackle in this paper is to find conditions on the parameters r, p and s in the template (3) under which the 1-D CNN described by (1)–(5) performs CCD in the sense of Definition 2. If $p - 1 \geq r + s$, the 1-D CNN can never perform CCD because $\boldsymbol{y}(t) = \boldsymbol{y}(0)$ holds for $t \geq 0$. Therefore we assume in the following that

$$0 < p - 1 < r + s\,.$$

It is known that under this condition the 1-D CNN has a global propagation behavior [12, Theorem 1].

3 Main Results

We first show a definition introduced by Hänggi [18] which will play an important role in our analysis.

Definition 3 *A 1-D CNN is said to be locally regular if*

$$|y_i(t)| < 1 \Rightarrow |y_{i-1}(t)| = |y_{i+1}(t)| = 1$$

holds for all i and all $t \geq 0$.

From a view point of image processing, the local regularity means that two adjacent pixels of the output $\boldsymbol{y}(t)$ never become gray simultaneously.

Now we are ready to present the main results of this paper.

Theorem 1 *If a 1-D CNN described by (1)–(5) satisfies*

$$|r - s| < p - 1 < r + s \tag{6}$$

$$g(p, r, s) \geq 0 \tag{7}$$

$$g(p, s, r) \geq 0 \tag{8}$$

where the function g is defined by

$$\begin{aligned} g(\alpha, \beta, \gamma) \triangleq (\beta + \gamma + \alpha - 1)^{\frac{1}{\alpha - 1}} \{\alpha(\alpha - 1 - \gamma) + \beta(\beta + \gamma - 1)\} \\ - (\beta + \gamma - \alpha + 1)^{\frac{1}{\alpha - 1}} \{\alpha(\alpha - 1 - \gamma) + \beta(\beta + \gamma + 1)\} \end{aligned}$$

then it is locally regular and its output $\boldsymbol{y}(t)$ satisfies

$$|y_i(t)| < 1 \;\Rightarrow\; y_{i-1}(t)\, y_{i+1}(t) = -1 \tag{9}$$

for $i = 1, 2, \ldots, n$ and all $t \geq 0$.

Theorem 2 *If a 1-D CNN described by (1)–(5) satisfies the same conditions as in Theorem 1, it performs CCD.*

Proofs of Theorems 1 and 2 are given in the next section.

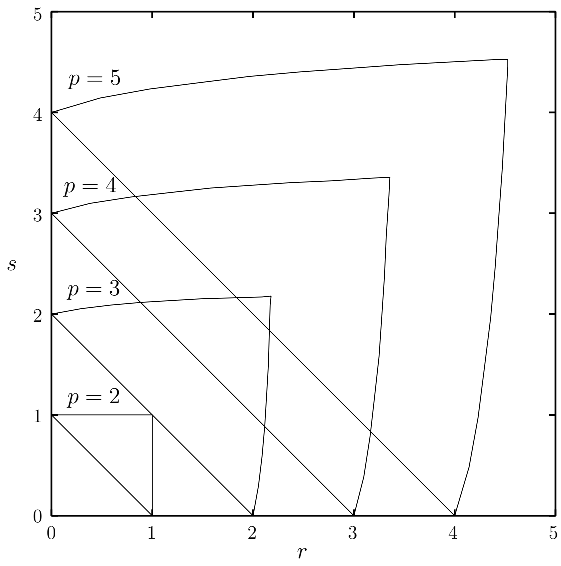

Fig. 2. Parameter regions in the (r, s)-plane for which (6)–(8) are satisfied with $p = 2$, 3, 4 and 5.

Figure 2 shows the parameter regions (the areas surrounded by the closed curves) in the (r, s)-plane for which the conditions (6)–(8) are satisfied with $p = 2$, 3, 4 and 5. It is clearly seen from this figure that the 1-D CNN can perform CCD for a wide range of parameter values. For example, in the case where $p = 5$ and $r = 4$, the parameter s can take any value between 0 and 4.

As a special case of Theorem 2, we can easily derive a simple sufficient condition as follows:

Corollary 1 *If a 1-D CNN described by (1)–(4) satisfies*

$$p = 2, \quad r \leq 1, \quad s \leq 1, \quad r + s > 1$$

then it performs CCD.

Proof. By substituting $p = 2$ into (6), (7) and (8), we have $|r - s| < 1 < r + s$, $s \leq 1$ and $r \leq 1$, respectively. The first inequality $|r - s| < 1$ is redundant because $r + s > 1$, $s \leq 1$ and $r \leq 1$ imply it. $\square$

Example 1 Let us consider a 1-D CNN with $n = 8$ and $A = [r, p, -s] = [1.8, 3, -0.8]$. As is easily seen from Fig.2, this CNN satisfies (6)–(8) and hence can perform CCD. Figure 3 shows the waveforms of $x_1(t), x_2(t), \ldots, x_8(t)$ for the initial condition $\boldsymbol{x}(0) = (1, -1, 1, -1, 1, 1, 1, -1)$, which were obtained by solving (1) numerically. The output $\boldsymbol{y}(t)$ starting from $(1, -1, 1, -1, 1, 1, 1, -1)$, which corresponds to the input image in Fig.1, certainly converges to $(-1, -1, -1, 1, -1, 1, -1, 1)$, which corresponds to the output image in Fig.1. Figure 4 shows the waveforms of $x_3(t)$, $x_4(t)$ and $x_5(t)$, from which we can confirm that both

$|x_3(t)| \geq 1$ and $|x_5(t)| \geq 1$ hold whenever $|x_4(t)| < 1$.

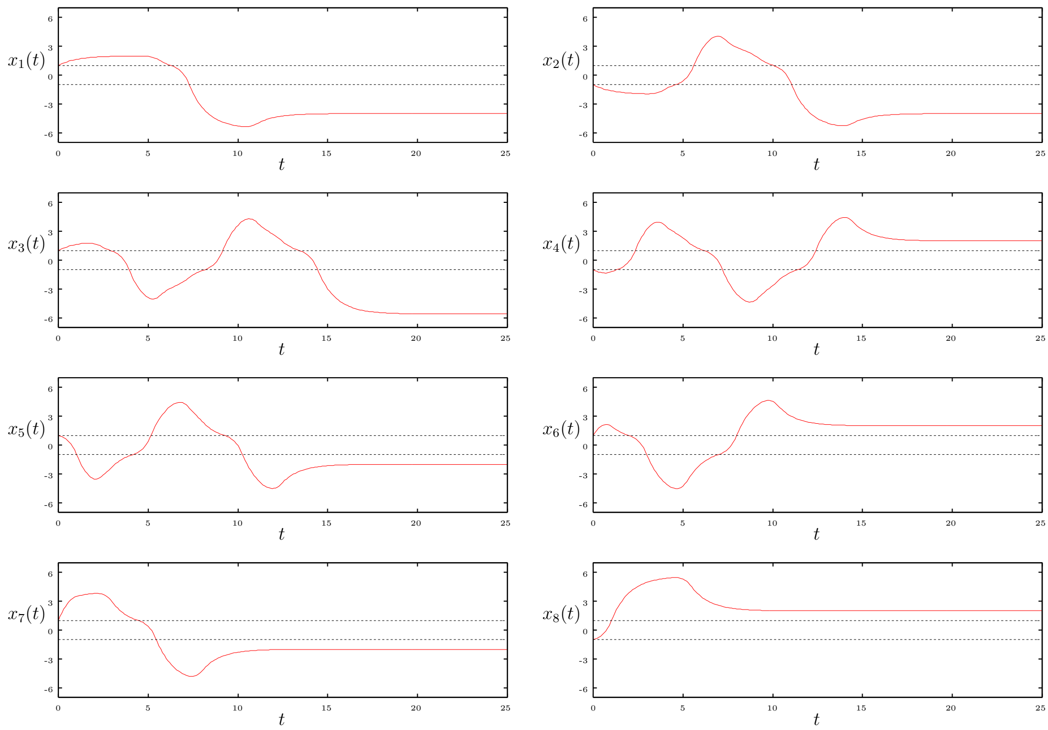

Fig. 3. Waveforms of $x_1(t), x_2(t), \ldots, x_8(t)$ generated by the 1-D CNN considered in Example 1.

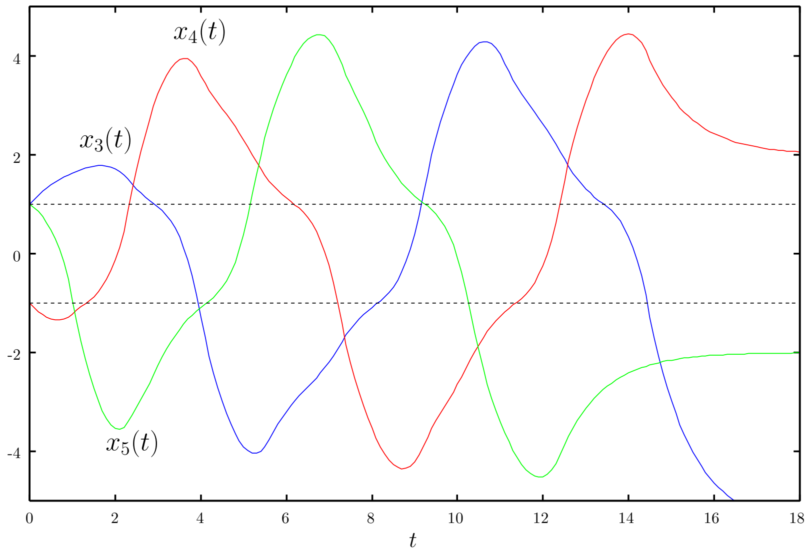

Fig. 4. Waveforms of $x_3(t), x_4(t)$ and $x_5(t)$ generated by the 1-D CNN considered in Example 1.

Example 2 Let us next consider a 1-D CNN with $n = 8$ and $A = [r, p, -s] = [2.6, 3, -0.8]$. It is easily verified from Fig.2 that this template does not satisfy (6)–(8). Figure 5 shows the waveforms of $x_3(t)$, $x_4(t)$ and $x_5(t)$ for the initial condition $\boldsymbol{x}(0) = (1, -1, 1, -1, 1, 1, 1, -1)$, from which we see that $|x_4(t)| < 1$ and $|x_5(t)| < 1$ hold simultaneously at $t \approx 0.6$ and that $|x_3(t)| < 1$ and $|x_4(t)| < 1$ hold simultaneously at $t \approx 1.1$. Therefore, this CNN is not locally regular. However, as shown in Fig. 6, the output $\boldsymbol{y}(t)$ starting from $(1, -1, 1, -1, 1, 1, 1, -1)$ converges to $(-1, -1, -1, 1, -1, 1, -1, 1)$ as in Example 1. This indicates that (6)–(8) are sufficient for CCD but may not be necessary.

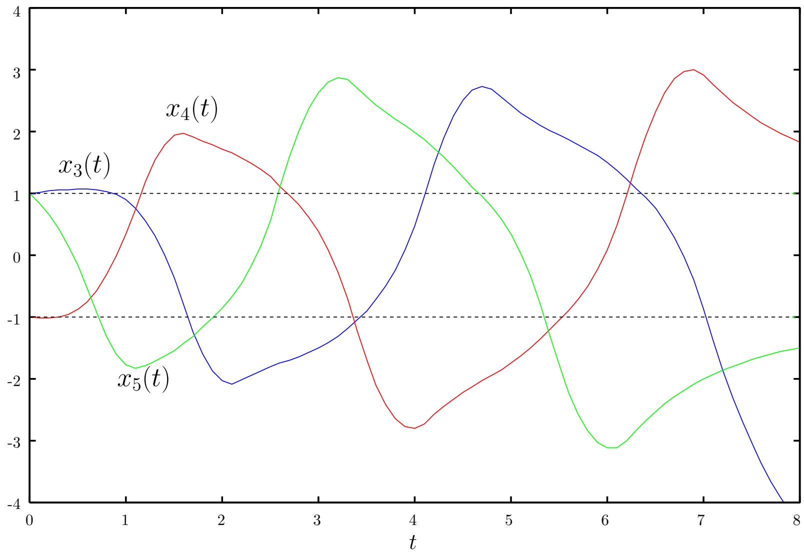

Fig. 5. Waveforms of $x_3(t), x_4(t)$ and $x_5(t)$ generated by the 1-D CNN considered in Example 2.

4 Proof of Theorem 1

We first give four lemmas which show some fundamental properties of 1-D CNNs satisfying (6)–(8) and play important roles in later discussions.

Lemma 1 *Let $\boldsymbol{y}(t)$ be an output trajectory of a CNN satisfying (6). If $(y_{i-1}(t_0), y_i(t_0)) \in \{(1,1), (-1,-1)\}$ holds for some $i \in \{1, 2, \ldots, n\}$ and some $t_0 \geq 0$ then $y_i(t)$ cannot become gray earlier than or at the same time as $y_{i-1}(t)$.*

Proof. Let us first consider the case where $(y_{i-1}(t_0),\ y_i(t_0)) = (1, 1)$. If $y_{i-1}(t) = 1$ and $x_i(t) = 1$, we have

$$\dot{x}_i(t) = -1 + r + p - s y_{i+1}(t) \geq p - 1 + r - s > 0$$

where the last inequality follows from (6). Therefore $y_i(t) = 1$ holds as long as $y_{i-1}(t) = 1$. Moreover, $y_{i-1}(t)$ and $y_i(t)$ cannot become gray at the same time. Let us next consider the case where $(y_{i-1}(t_0), y_i(t_0)) = (-1, -1)$. If

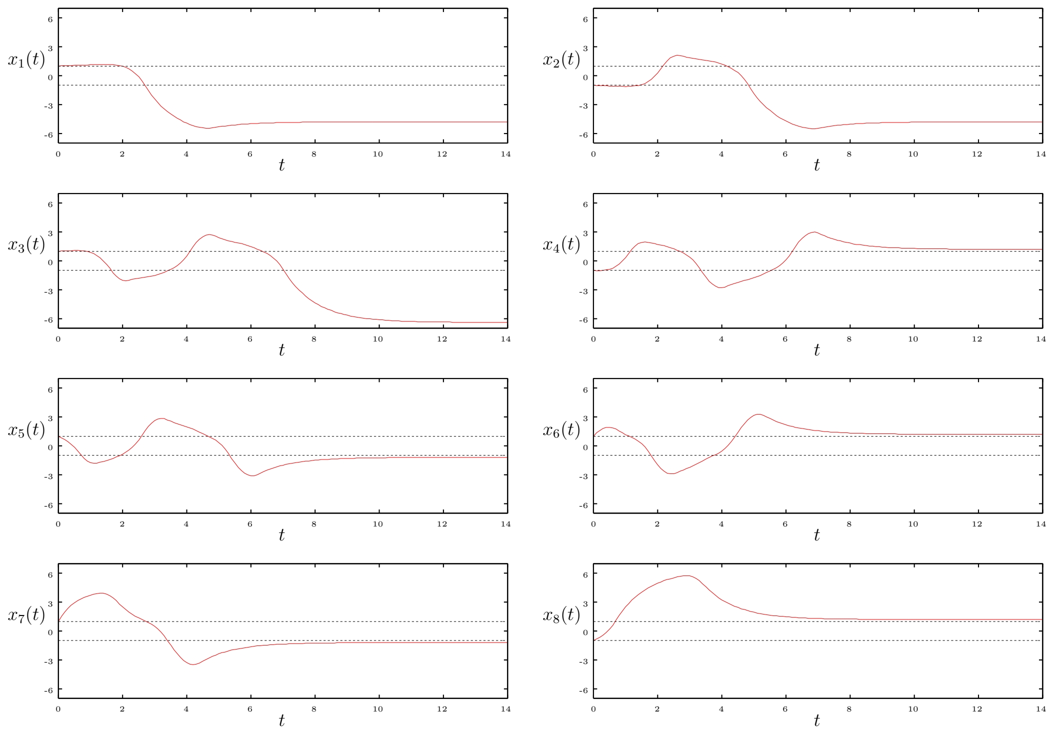

Fig. 6. Waveforms of $x_1(t), x_2(t), \ldots, x_8(t)$ generated by the 1-D CNN considered in Example 2.

$y_{i-1}(t) = -1$ and $x_i(t) = -1$, we have

$$\dot{x}_i(t) = 1 - r - p - sy_{i+1}(t) \leq -p + 1 - r + s < 0$$

where the last inequality follows from (6). Therefore $y_i(t) = -1$ holds as long as $y_{i-1}(t) = -1$. Moreover, $y_{i-1}(t)$ and $y_i(t)$ cannot become gray at the same time. □

Lemma 2 *Let $\boldsymbol{y}(t)$ be an output trajectory of a CNN satisfying (6). If $(y_i(t_0), y_{i+1}(t_0)) \in \{(1, -1), (-1, 1)\}$ holds for some $i \in \{1, 2, \ldots, n\}$ and some $t_0 \geq 0$ then $y_i(t)$ cannot become gray earlier than or at the same time as $y_{i+1}(t)$.*

We omit the proof of Lemma 2 because it is similar to that of Lemma 1.

Lemma 3 *Let $\boldsymbol{y}(t)$ be an output trajectory of a CNN satisfying (6)–(8). If $(y_{i-1}(t_0), y_i(t_0), y_{i+1}(t_0)) = (-1, 1, 1)$ holds for some $i \in \{1, 2, \ldots, n\}$ and some $t_0 \geq 0$ then $y_i(t)$ decreases monotonically until it reaches -1 at $t = t_1 \geq t_0 + T$ where*

$$T \triangleq \frac{1}{p-1} \log \left(\frac{r+s+p-1}{r+s-p+1} \right). \tag{10}$$

Moreover, $y_{i-1}(t)$ and $y_{i+1}(t)$ are constant for $t_0 \leq t \leq t_1$.

Proof. It follows from Lemmas 1 and 2 that neither $y_{i-1}(t)$ nor $y_{i+1}(t)$ become gray earlier than or at the same time as $y_i(t)$. Also, as far as $(y_{i-1}(t), y_i(t), y_{i+1}(t)) = (-1, 1, 1)$, $x_i(t)$ decreases monotonically until it reaches 1 because

$$\dot{x}_i(t) = -x_i(t) - r + p - s \le p - 1 - r - s < 0\,.$$

Hence there exists a $t^*(\ge t_0)$ such that $y_{i-1}(t^*) = -1$, $x_i(t^*) = 1$ and $y_{i+1}(t^*) = 1$. In the following, we assume without loss of generality that $t^* = 0$. Let $t_{\max} \triangleq \max\{\tau \,|\, y_{i-1}(t) = -1,\ |x_i(t)| \le 1,\ y_{i+1}(t) = 1,\ \forall t \in [0, \tau]\}$ Then $x_{i-1}(t)$, $x_i(t)$ and $x_{i+1}(t)$ obey the following differential equations:

$$\begin{aligned} \dot{x}_{i-1}(t) &= -x_{i-1}(t) + r y_{i-2}(t) - p - s x_i(t) && (11)\\ \dot{x}_i(t) &= (p-1)x_i(t) - r - s && (12)\\ \dot{x}_{i+1}(t) &= -x_{i+1}(t) + r x_i(t) + p - s y_{i+2}(t) && (13) \end{aligned}$$

for $0 \le t \le t_{\max}$. By solving (12) with the initial condition $x_i(0) = 1$, we can obtain an explicit formula of $x_i(t)$ $(0 \le t \le t_{\max})$ as follows:

$$x_i(t) = \left(1 - \frac{r+s}{p-1}\right) e^{(p-1)t} + \frac{r+s}{p-1} \triangleq a(t) \qquad (14)$$

It is easily seen from (6) that the function $a(t)$ defined above is monotone decreasing. Also, $a(t) = -1$ holds for

$$t = \frac{1}{p-1} \log\left(\frac{r+s+p-1}{r+s-p+1}\right) = T\,. \qquad (15)$$

In the following, we will show that $x_{i+1}(t) \ge 1$ holds for $0 \le t \le T$ if (6) and (7) are satisfied, and that $x_{i-1}(t) \le -1$ holds for $0 \le t \le T$ if (6) and (8) are satisfied.

From (13) we obtain a formal expression of $x_{i+1}(t)$ $(0 \le t \le t_{\max})$ as follows:

$$x_{i+1}(t) = x_{i+1}(0)\, e^{-t} + e^{-t} \int_0^t e^{\tau} (r x_i(\tau) + p - s y_{i+2}(\tau))\, \mathrm{d}\tau\,.$$

Since $x_{i+1}(0) \ge 1$ and $y_{i+2}(t) \le 1$, we have

$$x_{i+1}(t) \ge e^{-t} + e^{-t} \int_0^t e^{\tau} (r x_i(\tau) + p - s)\, \mathrm{d}\tau\,. \qquad (16)$$

Substituting (14) into (16), we can obtain a lower bound for $x_{i+1}(t)$ $(0 \leq t \leq t_{\max})$ as

$$\begin{aligned} x_{i+1}(t) \geq & -\left\{p-1+\frac{r}{p}-s+\frac{r(r+s)}{p}\right\} e^{-t} \\ & -\frac{r(r+s-p+1)}{p(p-1)} e^{(p-1)t}+p-s+\frac{r(r+s)}{p-1} \\ \triangleq & \; b_1(t) \end{aligned} \tag{17}$$

Now we will show that the function $b_1(t)$ defined above is greater than or equal to 1 for $0 \leq t \leq T$ if (6) and (7) hold. First we easily see that $b_1(0) = 1$. Next, since

$$\dot{b}_1(t) = \left\{p-1-s+\frac{r(r+s+1)}{p}\right\} e^{-t} - \frac{r(r+s-p+1)}{p} e^{(p-1)t},$$

we have from (6) that $\dot{b}(0) = p - 1 - s + r > 0$. Furthermore, since

$$p-1-s+\frac{r(r+s+1)}{p} > -r+\frac{r(r+s+1)}{p} = \frac{r(r+s-p+1)}{p} > 0$$

holds from (6), the second derivative of $b_1(t)$ satisfies

$$\begin{aligned} \ddot{b}_1(t) = & -\left\{p-1-s+\frac{r(r+s+1)}{p}\right\} e^{-t} - \frac{r(r+s-p+1)(p-1)}{p} e^{(p-1)t} \\ < & \; 0 \end{aligned}$$

for all t, which means that $b_1(t)$ is a concave function. From these properties, it suffices for us to show that $b_1(T) \geq 1$. Substituting (15) into (17), we have

$$\begin{aligned} b_1(T) = & -\left\{p-1-s+\frac{r(r+s+1)}{p}\right\} \left(\frac{r+s-p+1}{r+s+p-1}\right)^{\frac{1}{p-1}} \\ & -\frac{r(r+s+p-1)}{p(p-1)}+p-s+\frac{r(r+s)}{p-1} \end{aligned}$$

from which we obtain

$$\begin{aligned} b_1(T)-1 = & -\left\{p-1-s+\frac{r(r+s+1)}{p}\right\} \left(\frac{r+s-p+1}{r+s+p-1}\right)^{\frac{1}{p-1}} \\ & -\frac{r(r+s+p-1)}{p(p-1)}+p-1-s+\frac{r(r+s)}{p-1} \\ = & -\left\{p-1-s+\frac{r(r+s+1)}{p}\right\} \left(\frac{r+s-p+1}{r+s+p-1}\right)^{\frac{1}{p-1}} \\ & +p-1-s+\frac{r(r+s-1)}{p}. \end{aligned}$$

It is easily seen that the right-hand side is nonnegative if (7) is satisfied. Therefore we can conclude that $b_1(t) \geq 1$ holds for $0 \leq t \leq T$. Relationships among $x_i(t)$, $x_{i+1}(t)$ and $b_1(t)$ are depicted in Fig. 5.

From (11) we obtain the formal expression of $x_{i-1}(t)$ ($0 \leq t \leq t_{\max}$) as follows:

$$x_{i-1}(t) = x_{i-1}(0)\, e^{-t} + e^{-t} \int_0^t e^{\tau}(r y_{i-2}(\tau) - p - s x_i(\tau))\, \mathrm{d}\tau\,.$$

Since $x_{i-1}(0) \leq -1$ and $y_{i-2}(t) \leq 1$, we have

$$x_{i-1}(t) \leq -e^{-t} + e^{-t} \int_0^t e^{\tau}(r + p - s x_i(\tau))\, \mathrm{d}\tau\,. \tag{18}$$

Substituting (14) into (18), we can obtain an upper bound for $x_{i-1}(t)$ ($0 \leq t \leq t_{\max}$) as

$$\begin{aligned} x_{i-1}(t) \leq &\left\{ p - 1 + \frac{s}{p} - r + \frac{s(r+s)}{p} \right\} e^{-t} \\ &+ \frac{s(r+s-p+1)}{p(p-1)} e^{(p-1)t} - p + r - \frac{s(r+s)}{p-1} \\ &\triangleq b_2(t)\,. \end{aligned}$$

Here we should note that the function $b_2(t)$ defined above is identical to the one obtained by exchanging r and s in $-b_1(t)$. Thus we can immediately conclude that $b_2(t) \leq -1$ holds for $0 \leq t \leq T$ if (8) is satisfied. Relationships among $x_i(t)$, $x_{i-1}(t)$ and $b_2(t)$ are depicted in Fig. 5. $\square$

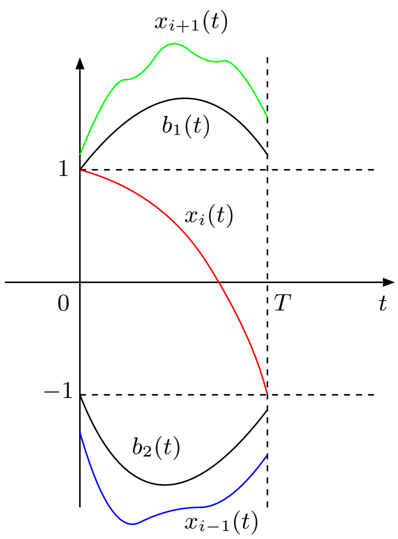

Fig. 7. Relationships among $x_i(t)$, $x_{i+1}(t)$, $x_{i-1}(t)$, $b_1(t)$ and $b_2(t)$ when (6)–(8) and $(y_{i-1}(0), x_i(0), y_{i+1}(0)) = (-1, 1, 1)$ are satisfied.

Lemma 4 *Let* $\boldsymbol{y}(t)$ *be an output trajectory of a CNN satisfying (6)–(8). If* $(y_{i-1}(t_0), y_i(t_0), y_{i+1}(t_0)) = (1, -1, -1)$ *holds for some* $i \in \{1, 2, \ldots, n\}$ *and* *some $t_0 \geq 0$ then $y_i(t)$ increases monotonically until it reaches 1 at $t = t_1 \geq t_0 + T$ where T is defined by (10). Moreover, $y_{i-1}(t)$ and $y_{i+1}(t)$ are constant for $t_0 \leq t \leq t_1$.*

Proof of Lemma 4 is omitted because it is similar to that of Lemma 3.

Now we are ready to prove Theorem 1. The proof is done by mathematical induction. We first show as the basis step that (9) holds for $i = 1, 2, \ldots, n$ and $0 \leq t \leq T$ where T is defined by (10). We next show as the induction step that if (9) holds for $i = 1, 2, \ldots, n$ and $0 \leq t \leq kT$ where k is any positive integer then it also holds for $i = 1, 2, \ldots, n$ and $kT \leq t \leq (k+1)T$.

Lemma 5 *Let $\boldsymbol{y}(t)$ be an output trajectory of a CNN satisfying (6)–(8). If $|y_{i-1}(t_0)| = |y_i(t_0)| = |y_{i+1}(t_0)| = 1$ holds for some $t_0 \geq 0$, then (9) holds for $t_0 \leq t \leq t_0 + T$.*

Proof. If $(y_{i-1}(t_0), y_i(t_0), y_{i+1}(t_0)) \in \{(-1, 1, 1), (1, -1, -1)\}$ then it is easily seen from Lemmas 3 and 4 that (9) holds for $t_0 \leq t \leq t_0+T$. If $(y_{i-1}(t_0), y_i(t_0), y_{i+1}(t_0)) \notin \{(-1, 1, 1), (1, -1, -1)\}$ then at least one of the following two conditions holds.

$$(y_{i-1}(t_0), y_i(t_0)) \in \{(1, 1), (-1, -1)\}$$
$$(y_i(t_0), y_{i+1}(t_0)) \in \{(1, -1), (-1, 1)\}$$

In the following, we will focus our attention on the case where $(y_{i-1}(t_0), y_i(t_0)) = (1, 1)$ and show that $y_i(t)$ is constant for $t_0 \leq t \leq t_0 + T$ which means that (9) holds for $t_0 \leq t \leq t_0+T$. Although other three cases will not be considered here, the same conclusion can be drawn in a similar way.

Let us assume without loss of generality that $t_0 = 0$. As far as $y_i(t) = 1$ is satisfied, $x_i(t)$ can be expressed as follows:

$$x_i(t) = x_i(0)e^{-t} + e^{-t} \int_0^t e^{\tau}(ry_{i-1}(\tau) + p - sy_{i+1}(\tau))\mathrm{d}\tau\,.$$

Note that $y_{i-1}(t)$ decreases most rapidly when $y_{i-2}(0) = -1$ and $x_{i-1}(0) = 1$ and in this case $y_i(t) = x_i(t) = a(t)$ holds for $0 \leq t \leq T$ where $a(t)$ is defined in (14). Note also that $x_i(0) \geq 1$ and $y_{i+1}(t) \leq 1$. From these observation, we have

$$x_i(t) \geq e^{-t} + e^{-t} \int_0^t e^{\tau}(ra(\tau) + p - s)\mathrm{d}\tau = b_1(t)\,.$$

Since $b_1(t) \geq 1$ holds for $0 \leq t \leq T$, as we have seen in the proof of Lemma 3, $y_i(t) = 1$ holds for $0 \leq t \leq T$. □

By substituting $t_0 = 0$ into Lemma 5, we immediately see that (9) holds for $i = 1, 2, \ldots, n$ and $0 \leq t \leq T$, which completes the proof of the basis step of the mathematical induction. Now we proceed to the induction step. Let j be any integer between 1 and n. The goal is to show that

$$|y_j(t)| < 1 \ \Rightarrow \ y_{j-1}(t)y_{j+1}(t) = -1 \tag{19}$$

holds for $kT \le t \le (k+1)T$ under the assumption that (9) holds for $i = 1, 2, \ldots, n$ and $0 \le t \le kT$. To do so, we divide the problem into the following five cases:

(1) $|y_{j-1}(kT)| = |y_j(kT)| = |y_{j+1}(kT)| = 1$
(2) $|y_j(kT)| < 1$
(3) $|y_{j-1}(kT)| < 1$, $|y_j(kT)| = 1$, $|y_{j+1}(kT)| < 1$
(4) $|y_{j-1}(kT)| < 1$, $|y_j(kT)| = |y_{j+1}(kT)| = 1$
(5) $|y_{j-1}(kT)| = |y_j(kT)| = 1$, $|y_{j+1}(kT)| < 1$

The first case has already been proved by Lemma 5. So we will consider in the following the remaining four cases.

Lemma 6 *Let $\boldsymbol{y}(t)$ be an output trajectory of a CNN satisfying (6)–(8). Suppose that (9) holds for $i = 1, 2, \ldots, n$ and $0 \le t \le kT$. If $|y_j(kT)| < 1$ then i) (19) holds for $kT \le t \le (k+1)T$ and ii) there exists a $t^* \in (kT, (k+1)T)$ such that $|y_j(t)| < 1$ for $kT \le t < t^*$ and $(y_{j-1}(t^*), y_j(t^*), y_{j+1}(t^*))$ is $(-1, -1, 1)$ or $(1, 1, -1)$.*

Proof. Let $t_1 = \max\{\tau \,|\, |y_j(\tau)| = 1,\ 0 \le \tau < kT\}$. Then the following two conditions apparently hold.

$$|y_j(t_1)| = |x_j(t_1)| = 1 \tag{20}$$
$$|y_j(t)| < 1, \quad t_1 < t \le kT \tag{21}$$

By the assumption and (21), we have

$$y_{j-1}(t)y_{j+1}(t) = -1, \quad t_1 < t \le kT\,.$$

Since $\boldsymbol{y}(t)$ is continuous, we have

$$y_{j-1}(t_1)y_{j+1}(t_1) = -1\,. \tag{22}$$

From (20) and (22), $(y_{j-1}(t_1), y_j(t_1), y_{j+1}(t_1))$ must be one of the following: $(-1, 1, 1)$, $(1, -1, -1)$, $(-1, -1, 1)$ or $(1, 1, -1)$. However, by Lemmas 1–4, (20) occurs if and only if $(y_{j-1}(t_1), y_j(t_1), y_{j+1}(t_1))$ is either $(-1, 1, 1)$ or $(1, -1, -1)$. In the former (latter) case, by Lemma 3 (Lemma 4), $y_j(t)$ decreases (increases) monotonically until it reaches -1 (1) at $t = t_1 + T$ while $y_{j-1}(t)$ and $y_{j+1}(t)$ are constant for $t_1 \le t \le t_1 + T$. Hence (19) holds for $t_1 \le t \le t_1 + T$. Note that $t_1 + T$ is greater than kT because otherwise $|y_j(t)| = 1$ holds for some $t \in (t_1, kT]$ which contradicts the assumption $|y_j(kT)| < 1$ or the definition of t_1. This completes the proof of the second statement. Since $|y_{j-1}(t_1 + T)| = |y_j(t_1 + T)| = |y_{j+1}(t_1 + T)| = 1$, it follows from Lemma 5 that (19) holds for

$t_1 + T \leq t \leq t_1 + 2T$ where $t_1 + 2T$ is greater than $(k+1)T$. Thus (19) holds for $kT \leq t \leq (k+1)T$. $\square$

Lemma 7 *Let $\boldsymbol{y}(t)$ be an output trajectory of a CNN satisfying (6)–(8). Suppose that (9) holds for $i = 1, 2, \ldots, n$ and $0 \leq t \leq kT$. If $|y_{j-1}(kT)| < 1$, $|y_j(kT)| = 1$ and $|y_{j+1}(kT)| < 1$ then (9) holds for $kT \leq t \leq (k+1)T$.*

Proof. By the assumption and Lemma 6, $|y_{j-1}(t)| = 1$ holds for some $t \in (kT, (k+1)T)$. Similarly, $|y_{j+1}(t)| = 1$ holds for some $t \in (kT, (k+1)T)$. Let $t_1 = \min\{\tau \,|\, |y_{j-1}(\tau)| = 1,\, \tau > kT\}$ and $t_2 = \min\{\tau \,|\, |y_{j+1}(\tau)| = 1,\, \tau > kT\}$. Then, by Lemma 6, we have

$$y_j(t) = y_j(kT) \in \{1, -1\}, \quad kT \leq t \leq \max\{t_1, t_2\} \tag{23}$$

which means (19) holds for $kT \leq t \leq \max\{t_1, t_2\}$. If $t_1 \leq t_2$ then $y_{j-1}(t_1)y_j(t_1) = -1$ follows from Lemma 6 and

$$y_{j-1}(t) = -y_j(kT) \in \{1, -1\}, \quad t_1 \leq t \leq t_2$$

follows from (23) and Lemma 2. Hence we have $|y_{j-1}(t_2)| = |y_j(t_2)| = |y_{j+1}(t_2)| = 1$. Then, by Lemma 5, (19) holds for $t_2 \leq t \leq t_2 + T$ where $t_2 + T$ is apparently greater than $(k+1)T$. If $t_1 > t_2$, on the other hand, then $y_j(t_2)y_{j+1}(t_2) = 1$ follows from Lemma 6 and

$$y_{j+1}(t) = y_j(kT) \in \{1, -1\}, \quad t_2 \leq t \leq t_1$$

follows from (23) and Lemma 1. Hence we have $|y_{j-1}(t_1)| = |y_j(t_1)| = |y_{j+1}(t_1)| = 1$. Then, by Lemma 5, (19) holds for $t_1 \leq t \leq t_1 + T$ where $t_1 + T$ is apparently greater than $(k+1)T$. $\square$

Lemma 8 *Let $\boldsymbol{y}(t)$ be an output trajectory of a CNN satisfying (6)–(8). Suppose that (9) holds for $i = 1, 2, \ldots, n$ and $0 \leq t \leq kT$. If $|y_{j-1}(kT)| < 1$, $|y_j(kT)| = 1$ and $|y_{j+1}(kT)| = 1$ then (19) holds for $kT \leq t \leq (k+1)T$.*

Proof. Let $t_1 = \min\{\tau \,|\, |y_{j-1}(\tau)| = 1,\ \tau > kT\}$. Then, from the assumption and Lemma 6,

$$|y_{j-1}(t)| < 1 \ \Rightarrow \ y_{j-2}(t)y_j(t) = -1,\ kT \leq t \leq (k+1)T$$

and

$$|y_j(t)| = 1, \quad kT \leq t \leq t_1 \tag{24}$$

hold. If $|y_{j+1}(t_1)| = 1$ then it follows from Lemma 5 that (19) holds for $t_1 \leq t \leq t_1 + T$. Thus we will focus our attention on the case where $|y_{j+1}(t_1)| < 1$.

Let us assume for the moment that

$$|y_{j+1}(t)| < 1 \ \Rightarrow \ y_j(t)y_{j+2}(t) = -1,\ kT \leq t \leq t_1 \tag{25}$$

holds. Then, since $|y_{j-1}(t_1-\varepsilon)| < 1$, $|y_j(t_1-\varepsilon)| = 1$ and $|y_{j+1}(t_1-\varepsilon)| < 1$ hold for sufficiently small $\varepsilon > 0$, it follows from Lemma 7 that (19) holds for $t_1-\varepsilon \le t \le t_1 - \varepsilon + T$. This and (24) imply that (19) holds for $kT \le t \le (k+1)T$. In the following, we will show that (25) indeed holds. Consider first the case where $|y_{j+2}(kT)| = 1$. In this case, by Lemma 5, (25) holds. Consider next the case where $|y_{j+2}(kT)| < 1$. Let $t_2 = \min\{\tau \mid |y_{j+2}(\tau)| = 1,\ \tau > kT\}$. Then it is apparent that $|y_{j+1}(t)| = 1$ for $kT \le t \le t_2$. In addition, we have $kT < t_2 < t_1$ because otherwise $y_{j+1}(t)$ must satisfy $|y_{j+1}(t)| = 1$ for $kT \le t \le t_1$ which contradicts $|y_{j+1}(kT)| < 1$. Since $|y_j(t_2)| = |y_{j+1}(t_2)| = |y_{j+2}(t_2)| = 1$, we can conclude from Lemma 5 that (25) holds. □

Lemma 9 *Let $\boldsymbol{y}(t)$ be an output trajectory of a CNN satisfying (6)–(8). Suppose that (9) holds for $i = 1, 2, \ldots, n$ and $0 \le t \le kT$. If $|y_{j-1}(kT)| = 1$, $|y_j(kT)| = 1$ and $|y_{j+1}(kT)| < 1$ then (19) holds for $kT \le t \le (k+1)T$.*

Proof of Lemma 9 is omitted because it is similar to that of Lemma 8.

5 Proof of Theorem 2

Proof of Theorem 2 is done in two steps. We first show that the output $\boldsymbol{y}(t)$ always converges to a binary vector and that $\lim_{t\to\infty} \boldsymbol{y}(t)$ satisfies the conditions 2), 3) and 4) in Definition 1. We next show that $\lim_{t\to\infty} \boldsymbol{y}(t)$ also satisfies the condition 1) in Definition 1.

Lemma 10 *Let $\boldsymbol{y}(t)$ be an output trajectory of a CNN satisfying (6)–(8). Suppose that $y_{i+1}(t) = \alpha \in \{1, -1\}$ for all $t \ge t_0$ where $i \in \{1, 2, \ldots, n\}$. Then the following statements are true: i) If $\alpha = 1$ then there exists a $t_1(\ge t_0)$ such that $y_i(t) = -1$ for all $t \ge t_1$, ii) If $\alpha = -1$ and $y_1(t_0) = y_2(t_0) = \cdots = y_i(t_0) = -1$ then $y_1(t), y_2(t), \ldots, y_i(t)$ remain constant for all $t \ge t_0$, and iii) If $\alpha = -1$ and $y_j(t_0) > -1$ holds for some $j \in \{1, 2, \ldots, i\}$ then there exists a $t_1(\ge t_0)$ such that $y_i(t) = 1$ for all $t \ge t_1$.*

Proof. Let us begin with the first statement. If $y_i(t_0) = -1$ then, by Lemma 2, $y_i(t) = -1$ holds for all $t \ge t_0$. If $|y_i(t_0)| < 1$ then, by Lemma 6, $y_i(t)$ becomes -1 within a finite period of time and, by Lemma 2, $y_i(t)$ remains constant thereafter. If $y_i(t_0) = 1$ then by letting $j^* = \max\{j \mid y_j(t_0) = -1, 0 \le j \le i-1\}$ we have

$$(y_{j^*}(t_0), y_{j^*+1}(t_0), y_{j^*+2}(t_0), \ldots, y_i(t_0)) = (-1, \gamma, 1, \ldots, 1)$$

where $-1 < \gamma \le 1$. By Lemmas 3 and 6, $y_{j^*+1}(t)$ decreases monotonically and reaches -1 within a finite period of time. During this period, $y_{j^*+2}(t), y_{j^*+3}(t), \ldots, y_i(t)$ are constant due to Lemmas 1 and 6. After that $y_{j^*+2}(t)$ starts decreasing and reaches -1 within a finite period of time. During this period,

$y_{j^*+3}(t), \ldots, y_i(t)$ are constant due to Lemmas 1 and 6. This process is repeated and finally $y_i(t)$ starts decreasing and reaches -1 within a finite period of time. Once $y_i(t)$ becomes -1, it remains constant thereafter due to Lemma 2.

Let us next prove the second statement. If $y_0(t_0) = y_1(t_0) = \cdots = y_i(t_0) = -1$ then, by Lemma 1, $y_1(t), y_2(t), \ldots, y_i(t)$ are constant for all $t \geq t_0$.

Let us finally prove the third statement. If $y_i(t_0) = 1$ then, by Lemma 2, $y_i(t) = 1$ holds for all $t \geq t_0$. If $|y_i(t_0)| < 1$ then, by Lemma 6, $y_i(t)$ becomes 1 within a finite period of time and, by Lemma 2, $y_i(t)$ remains constant thereafter. If $y_i(t_0) = -1$ then $y_j(t) > -1$ holds for some $j \in \{1, 2, \ldots, i-1\}$ and, by Theorem 1, there exists at least one $j \in \{1, 2, \ldots, i-1\}$ satisfying $y_j(t_0) = 1$. Let $j^* = \max\{j \mid y_j(t_0) = 1, 0 \leq j \leq i-1\}$. Then we have

$$(y_{j^*}(t_0), y_{j^*+1}(t_0), y_{j^*+2}(t_0), \ldots, y_i(t_0)) = (1, \gamma, -1, \ldots, -1)$$

where $-1 \leq \gamma < 1$. By Lemmas 4 and 6, $y_{j^*+1}(t)$ increases monotonically and reaches 1 within a finite period of time. During this period, $y_{j^*+2}(t), y_{j^*+3}(t), \ldots, y_i(t)$ are constant due to Lemmas 1 and 6. After that $y_{j^*+2}(t)$ starts increasing and reaches 1 within a finite period of time. During this period, $y_{j^*+3}(t), \ldots, y_i(t)$ are constant due to Lemmas 1 and 6. This process is repeated and finally $y_i(t)$ starts increasing and reaches 1 within a finite period of time. Once $y_i(t)$ becomes 1, by Lemma 2, it remains 1 thereafter. □

Lemma 11 *If a CNN satisfies (6)–(8) then its output $\boldsymbol{y}(t)$ always converges to a binary vector which satisfies the conditions 2), 3) and 4) of Definition 1.*

Proof. Since $y_{n+1}(t)$ is fixed to -1 for all $t \geq 0$, we see from Lemma 10 that $y_1(t) = y_2(t) = \cdots = y_n(t) = -1$ for all $t \geq 0$ if $y_1(0) = y_2(0) = \cdots = y_n(0) = -1$ and there exists a $t_1(\geq 0)$ such that $y_n(t) = 1$ for all $t \geq t_1$ otherwise. In the latter case, we see again from Lemma 10 that there exists a $t_2(\geq t_1)$ such that $y_{n-1}(t) = -1$ for all $t \geq t_2$. By applying this argument to $y_{n-2}(t), y_{n-3}(t), \ldots, y_1(t)$, we can conclude that $\boldsymbol{y}(t)$ becomes a constant binary vector which satisfies the conditions 2), 3) and 4) of Definition 1 within a finite period of time. □

The final step to prove Theorem 2 is to show that the number of connected components in $\lim_{t\to\infty} \boldsymbol{y}(t)$ is equal to that in $\boldsymbol{y}(0)$. We will do this by using not the number of connected components but the number of transitions defined below.

Definition 4 *An index $i \in \{0, 1, \ldots, n\}$ is said to be transitional in $\boldsymbol{y}(t)$ if $|y_i(t)| = 1$ and $y_i(t)y_{i+1}(t) < 1$. The number of transitions in $\boldsymbol{y}(t)$, which is denoted by $N_{\mathrm{T}}(t)$, is defined to be the number of transitional indices.*

For example, if $\boldsymbol{y}(t) = (y_1(t), y_2(t), \ldots, y_8(t)) = (0.3, 1, -1, -1, -0.5, 1, 1, -1)$ then $N_{\mathrm{T}}(t) = 4$ because $i = 0$, 2, 4 and 7 are transitional indices.

An important property of the number of transitions in $\boldsymbol{y}(t)$ is that it is equal to twice the number of connected components in $\boldsymbol{y}(t)$ when $\boldsymbol{y}(t)$ is a binary vector. Therefore, it suffices for us to show that $\lim_{t\to\infty} N_{\mathrm{T}}(t)$ is equal to $N_{\mathrm{T}}(0)$. This is indeed true as shown in the following lemma.

Lemma 12 *If a CNN satisfies (6)–(8) then the number of transitions $N_{\mathrm{T}}(t)$ is invariant for all $t \geq 0$.*

Proof. The value of $N_{\mathrm{T}}(t)$ can change only when at least one component of $\boldsymbol{y}(t)$ becomes gray from black/white or becomes black/white from gray. Let us suppose that this occurs at $t = t^*$ for the set of indices $M \subseteq \{1, 2, \ldots, n\}$. If neither i nor $i+1$ belongs to M then the status of the index i, that is, whether or not i is transitional in $\boldsymbol{y}(t)$, does not change around $t = t^*$. So the number of transitional indices among those i such that neither i nor $i+1$ belongs to M does not change around $t = t^*$. If either i or $i+1$ belongs M then the status of the index i may change, but the total number of transitional indices among those i such that either i or $i+1$ belongs M does not change around $t = t^*$ as shown below.

(1) If $y_i(t)$ becomes gray from black at $t = t^*$ then $y_{i-1}(t)$ and $y_{i+1}(t)$ must be -1 and 1, respectively, at and just after $t = t^*$. Thus $i-1$ is transitional and i is not transitional in $\boldsymbol{y}(t)$ at and just after $t = t^*$.

(2) If $y_i(t)$ becomes gray from white at $t = t^*$ then $y_{i-1}(t)$ and $y_{i+1}(t)$ must be 1 and -1, respectively, at and just after $t = t^*$. Thus $i-1$ is transitional and i is not transitional in $\boldsymbol{y}(t)$ at and just after $t = t^*$.

(3) If $y_i(t)$ becomes black from gray at $t = t^*$ then $y_{i-1}(t)$ and $y_{i+1}(t)$ must be 1 and -1, respectively, just before and at $t = t^*$ Thus $i-1$ is transitional and i is not transitional in $\boldsymbol{y}(t)$ just before $t = t^*$, while $i-1$ is not transitional and i is transitional in $\boldsymbol{y}(t)$ at $t = t^*$.

(4) If $y_i(t)$ becomes white from gray at $t = t^*$ then $y_{i-1}(t)$ and $y_{i+1}(t)$ must be -1 and 1, respectively, just before and at $t = t^*$. Thus $i-1$ is transitional and i is not transitional in $\boldsymbol{y}(t)$ just before $t = t^*$, while $i-1$ is not transitional and i is transitional in $\boldsymbol{y}(t)$ at $t = t^*$.

From these observations, we can conclude that the number of transitions $N_{\mathrm{T}}(t)$ does not change around $t = t^*$. This completes the proof. □

6 Concluding Remarks

We have derived sufficient conditions for 1-D CNNs to perform CCD by restricting ourselves to locally regular 1-D CNNs. However, the local regularity may not be necessarily required for CCD as shown in Example 2. Thus if we remove this restriction, that is, if we allow adjacent cells to become gray at the same time, then milder conditions may be obtained. Exploring this possibility is one of the future problems.

Also, properties of the solutions of the inequalities (7) and (8) are not well understood except in some special cases [21]. In fact, parameter regions in Fig.2 were obtained by just solving (7) and (8) numerically. Further analysis will give us deeper understanding of these two inequalities.

Finally, we should note that the conditions (6)–(8) do not imply the complete stability of 1-D CNNs because in this paper we have considered only those state trajectories $\boldsymbol{x}(t)$ with $|x_i(0)| \geq 1$ for all i. Complete stability analysis is also one of the future problems.

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