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On Topology of Networked Multi-Agent Systems for Fast Consensus

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Abstract—The rate of convergence of a consensus algorithm for networked multi-agent systems is determined by the second smallest eigenvalue of the graph Laplacian. In this paper, we consider the problem of finding undirected graphs maximizing the second smallest eigenvalue of the Laplacian for the given number of nodes and edges. We show that under certain conditions the second smallest eigenvalue of the Laplacian is maximized for some well-known classes of graphs such as the cycle graph, the star graph and the complete bipartite graph.

1. Introduction

A multi-agent system is a system composed of many agents cooperating with each other to solve a certain problem. In recent years, the consensus problem for multi-agent systems has attracted considerable attention from the engineering community [1, 2]. We say that a consensus is reached when the states of all agents converge to the same value, e.g., the average of the initial states. Consensus problem is thus closely related to synchronization of coupled oscillators, flocking theory and distributed sensor fusion in sensor networks [1].

Olfati-Saber and Murray [2] proposed a consensus algorithm for multi-agent systems in which the time derivative of the state of each agent is determined by the states of other agents that can interact with it. They proved under certain mild assumptions that the system reaches the average consensus for any initial state. They also showed that the speed of convergence of the system is proportional to the second smallest eigenvalue of the graph Laplacian.

In this paper, we consider the problem of finding a graph that maximizes the second smallest eigenvalue of the Laplacian among all graphs with the prescribed number of nodes and edges. This is a fundamental problem closely related to the consensus algorithm proposed by Olfati-Saber and Murray [2]. First we focus our attention on graphs composed of 5 or 6 nodes and find a graph maximizing the second smallest eigenvalue of the Laplacian. Next we prove that 1) the cycle graph maximizes the second smallest eigenvalue when $n = m \leq 6$ where n and m are the number of nodes and edges, respectively, 2) a graph obtained by adding an edge to the star graph maximizes the second smallest eigenvalue when $n = m \geq 6$, 3) the cycle graph locally maximizes the second smallest eigenvalue for any $n \geq 3$, 4) the complete bipartite graph $K_{a,n-a}$ maximizes the second smallest eigenvalue when $a - 2a^2/n < 1$ and 5) the complete bipartite graph locally maximizes the second smallest eigenvalue.

2. Consensus Algorithm

In a multi-agent system, each agent interacts with all or part of other agents and thus these interactions are often represented by a graph of which each node corresponds to an agent. Throughout this paper, we assume that interactions between agents are time-invariant and symmetric[1] for simplicity. Under this assumption, the interaction topology of a multi-agent system can be represented by a simple undirected graph $G = (V, E)$ where $V = \{1, 2 \ldots, n\}$ is the set of nodes and E is the set of edges, that is, the set of unordered pairs of two different nodes. A pair $\{i, j\}$ is a member of E if and only if there is interaction between agents i and j.

Let $A = (a_{i,j}) \in \{0, 1\}^{n\times n}$ be the adjacency matrix of a simple undirected graph $G = (V, E)$. Note that A is a symmetric matrix. Let $D = \mathrm{diag}(d_1, \ldots, d_n)$ be the degree matrix of G with elements $d_i = \sum_{j\neq i} a_{ij}$ for $i = 1, 2, \ldots, n$. Then the graph Laplacian of G is defined by

$$L = D - A . \qquad (1)$$

By the definition of the graph Laplacian, it is apparent that $L\mathbf{1} = \mathbf{0} = 0 \cdot \mathbf{1}$ holds where $\mathbf{1}$ is an n-dimensional column vector of ones. This means that 0 is an eigenvalue of L and its associated eigenvector is $\mathbf{1}$. Also, it is easily seen from Gershgorin's theorem that all eigenvalues of L is nonnegative. From these facts, we can immediately conclude that the smallest eigenvalue of L is always 0. On the other hand, it is well known that the second smallest eigenvalue of L is nonzero if and only if G is connected.

Consensus algorithm proposed by Olfati-Saber and Murray [1] is described by the set of differential equations:

$$\dot{x}_i(t) = \sum_{j=1}^{n} a_{ij}(x_j(t) - x_i(t)), \quad i = 1, 2, \ldots, n \qquad (2)$$

[1]It is of course possible to consider nonsymmetric interactions [1, 2].

where $x_i(t)$ is the state of agent i at time t, $\dot{x}_i(t)$ is the time derivative of $x_i(t)$, and $a_{i,j}$ is the (i, j) entry of the adjacency matrix A of the graph G for the multi-agent system. By using the Laplacian L of G, Eq.(2) can be rewritten as

$$\dot{x}(t) = -Lx(t) \tag{3}$$

where $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$. The consensus algorithm (3) reaches an average-consensus if the corresponding graph G is connected [1]. More strictly speaking, $x_i(t)$ asymptotically converges to the average $\sum_{j=1}^{n} x_j(0)/n$ of n initial states for all i. In addition, the speed of convergence is proportional to the second smallest eigenvalue of L. In algebraic graph theory, the second smallest eigenvalue of the graph Laplacian is called the algebraic connectivity [3]. Therefore, we hereafter use this terminology.

The algebraic connectivity of graphs considerably differ by their topology even though the number of nodes and edges are the same. For example, the speed of convergence of the algorithm (2) for a system represented by the star graph is ten times faster than that by the path graph.

3. Undirected Graphs Maximizing the Algebraic Connectivity

3.1. Problem Formulation

Let $\mathcal{G}_{n,m}$ be the set of simple undirected graphs composed of n nodes and m edges. Let $\lambda_1(G), \lambda_2(G), \ldots, \lambda_n(G)$ be n eigenvalues of the Laplacian of $G \in \mathcal{G}_{n,m}$. We assume without loss of generality that eigenvalues are sorted in ascending order as $0 = \lambda_1(G) \leq \lambda_2(G) \leq \ldots \leq \lambda_n(G)$. Then $\lambda_2(G)$ is the algebraic connectivity of G.

Definition 1 *A graph $G \in \mathcal{G}_{n,m}$ is said to be an algebraic connectivity maximizing graph in $\mathcal{G}_{n,m}$ if it satisfies $\lambda_2(G) \geq \lambda_2(G')$ for all $G' \in \mathcal{G}_{n,m}$.*

Definition 2 *The set of all simple undirected graphs obtained by rewiring one edge in $G \in \mathcal{G}_{n,m}$ is called the neighborhood of G in $\mathcal{G}_{n,m}$ and denoted by $\mathcal{N}_{n,m}(G)$.*

Definition 3 *A graph $G \in \mathcal{G}_{n,m}$ is said to be an algebraic connectivity locally maximizing graph in $\mathcal{G}_{n,m}$ if it satisfies $\lambda_2(G) \geq \lambda_2(G')$ for all $G' \in \mathcal{N}_{n,m}(G)$.*

The problem we tackle in this paper is to find algebraic connectivity maximizing graphs and algebraic connectivity locally maximizing graphs for the given number of nodes and edges.

3.2. Graphs Composed of 5 or 6 Nodes

Let us first consider the case where n is either 5 or 6. In this case, all algebraic connectivity maximizing graphs can be found by a brute-force search. Results for $n = 5$ are summarized in Fig. 1 where algebraic connectivity maximizing graphs for $\mathcal{G}_{5,4}, \mathcal{G}_{5,5}, \ldots, \mathcal{G}_{5,10}$ and the values of the algebraic connectivity of those graphs are shown. Similarly, results for $n = 6$ are summarized in Fig. 2. Note that two graphs are presented in Figs. 1 (d), 2 (b) and 2 (g). This means that algebraic connectivity maximizing graph in $\mathcal{G}_{n,m}$ is not uniquely determined in general.

3.3. Trees

Let $\delta(G)$ be the minimum degree of a graph G. Then the following lemma holds.

Lemma 1 ([3]) *If G is not a complete graph then $\lambda_2(G) \leq \delta(G)$.*

If G is a tree, that is, the number of edges is less than that of nodes by one, then $\delta(G) = 1$. By Lemma 1, we have

$$\lambda_2(G) \leq 1. \tag{4}$$

The graph with $E = \{\{1, 2\}, \{1, 3\}, \ldots, \{1, n\}\}$ is called the star graph and denoted by S_n. Concerning eigenvalues of the Laplacian of S_n, the following result is known.

Lemma 2 ([4]) *Eigenvalues of the Laplacian of the star graph S_n are given by*

$$\lambda_i(S_n) = \begin{cases} 0, & i = 1 \\ 1, & i = 2, 3, \ldots, n-1 \\ n, & i = n \end{cases}$$

By (4) and Lemma 2, we get following theorem.

Theorem 1 ([5]) *The star graph S_n is a algebraic connectivity maximizing graph in $\mathcal{G}_{n,n-1}$.*

This result is also confirmed from Figs. 1 (a) and 2 (a).

3.4. Case where $m = n$

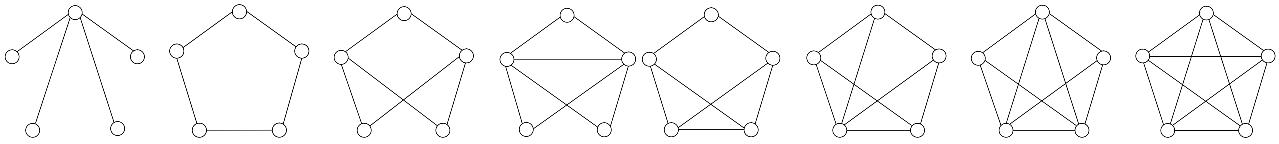
We next consider graphs having n nodes and n edges. An example is the graph with $E = \{\{1, 2\}, \{2, 3\}, \ldots, \{n-1, n\}, \{n, 1\}\}$ which is called the cycle graph and denoted by C_n. Eigenvalues of the Laplacian of the cycle graph are known as follows.

Lemma 3 ([4]) *Eigenvalues of the Laplacian of the cycle graph C_n are given by*

$$\lambda_k(C_n) = 2\left(1 - \cos\frac{2\pi(k-1)}{n}\right), \quad k = 1, 2, \ldots, n\,.$$

By Lemmas 1 and 3, we get the following theorem.

Theorem 2 *The cycle graph C_n is an algebraic connectivity maximizing graph in $\mathcal{G}_{n,n}$ when $3 \leq n \leq 6$.*



(a) $m = 4$ (b) $m = 5$ (c) $m = 6$ (d) $m = 7$ (e) $m = 8$ (f) $m = 9$ (g) $m = 10$

Figure 1: Algebraic connectivity maximizing graphs for $n = 5$. The values of algebraic connectivity are (a) 1, (b) 1.381, (c) 2, (d) 2, (e) 3, (f) 3 and (g) 5.

Proof: Let G be any graph in $\mathcal{G}_{n,n}$. Then the average degree of G is 2 and there are two possible cases: 1) all nodes have degree 2 and 2) not all nodes have degree 2. In the former case, G must be the cycle graph C_n. The algebraic connectivity of C_n is given by Lemma 3 as

$$\lambda_2(C_n) = 2\left(1 - \cos\frac{2\pi}{n}\right) \tag{5}$$

which is greater than 1 when $3 \leq n \leq 6$. On the other hand, in the latter case, since at least one node has degree 1, we have from Lemma 1 that $\lambda_2(G) \leq \delta(G) = 1$. Therefore, the cycle graph C_n is an algebraic connectivity maximizing graph in $\mathcal{G}_{n,n}$ for $3 \leq n \leq 6$. □

Let G be any graph in $\mathcal{G}_{n,m}$ and $G' \in \mathcal{G}_{n,m+1}$ be any graph obtained by adding one edge to G. The following lemma shows the relationship among eigenvalues of the Laplacians of G and G'.

Lemma 4 ([5]) *Let $G' \in \mathcal{G}_{n,m+1}$ be a graph obtained by adding an edge to $G \in \mathcal{G}_{n,m}$. Then we have*

$$\lambda_1(G) \leq \lambda_1(G') \leq \lambda_2(G) \leq \ldots \lambda_n(G) \leq \lambda_n(G').$$

By Lemmas 1, 2, 3 and 4, we have the following theorem.

Theorem 3 *A graph obtained by adding an edge to the star graph S_n is an algebraic connectivity maximizing graph in $\mathcal{G}_{n,n}$ when $n \geq 6$.*

Proof: By Lemmas 2 and 4, the algebraic connectivity of any graph obtained by adding an edge to the star graph is 1. We show that $\lambda_2(G) \leq 1$ for any $G \in \mathcal{G}_{n,n}$ ($n \geq 6$) in the following. Since the average degree of G is 2, there are two possible cases: 1) all nodes have degree 2 and 2) not all nodes have degree 2. In the former case, G must be the cycle graph C_n. By this fact and Eq.(5), $\lambda_2(G)$ is not greater than 1 for all $n \geq 6$. On the other hand, in the latter case, $\delta(G) = 1$ because at least one node have degree 1. By Lemma 1, we have $\lambda_2(G) \leq \delta(G) = 1$. Therefore, when $n \geq 6$, any graph obtained by adding an edge to the star graph is an algebraic connectivity maximizing graph in $\mathcal{G}_{n,n}$. □

The graph with $E = \{\{1,2\},\{2,3\},\ldots,\{n-1,n\}\}$ is called the path graph and denoted by P_n. Concerning eigenvalues of the Laplacian of P_n, the following result is known.

Lemma 5 ([4]) *Eigenvalues of the Laplacian of the path graph P_n are given by*

$$\lambda_k(P_n) = 2\left(1 - \cos\frac{\pi(k-1)}{n}\right), \quad k = 1, 2, \ldots, n.$$

Theorem 4 *The cycle graph is an algebraic connectivity locally maximizing graph in $\mathcal{G}_{n,n}$.*

Proof: Any graph obtained by removing an edge from the cycle graph C_n is isomorphic to the path graph P_n. Let G be any graph belonging to the neighborhood $\mathcal{N}_{n,n}(C_n)$ of C_n in $\mathcal{G}_{n,n}$. Then G is isomorphic to a graph obtained by adding an edge to the path graph P_n. By Lemma 4, we have

$$\lambda_2(P_n) \leq \lambda_2(G) \leq \lambda_3(P_n). \tag{6}$$

Also, by Lemmas 3 and 5, we have

$$\lambda_3(P_n) = 2\left(1 - \cos\frac{2\pi}{n}\right) = \lambda_2(C_n). \tag{7}$$

Therefore, we have from (6) and (7) that $\lambda_2(G) \leq \lambda_2(C_n)$. Since G is any graph in $\mathcal{N}_{n,n}(C_n)$, we conclude that the cycle graph C_n is an algebraic connectivity locally maximizing graph in $\mathcal{G}_{n,n}$. □

3.5. Complete Bipartite Graphs

A bipartite graph is a graph whose vertices can be divided into two disjoint sets V_1 and V_2 such that every edge has one end in V_1 and the other end in V_2. A complete bipartite graph is a bipartite graph in which there is an edge between any pair of nodes $i_1 \in V_1$ and $i_2 \in V_2$. The complete bipartite graph with partitions of size $|V_1| = a$ and $|V_2| = n - a$ ($a \leq n - a$) is denoted $K_{a,n-a}$.

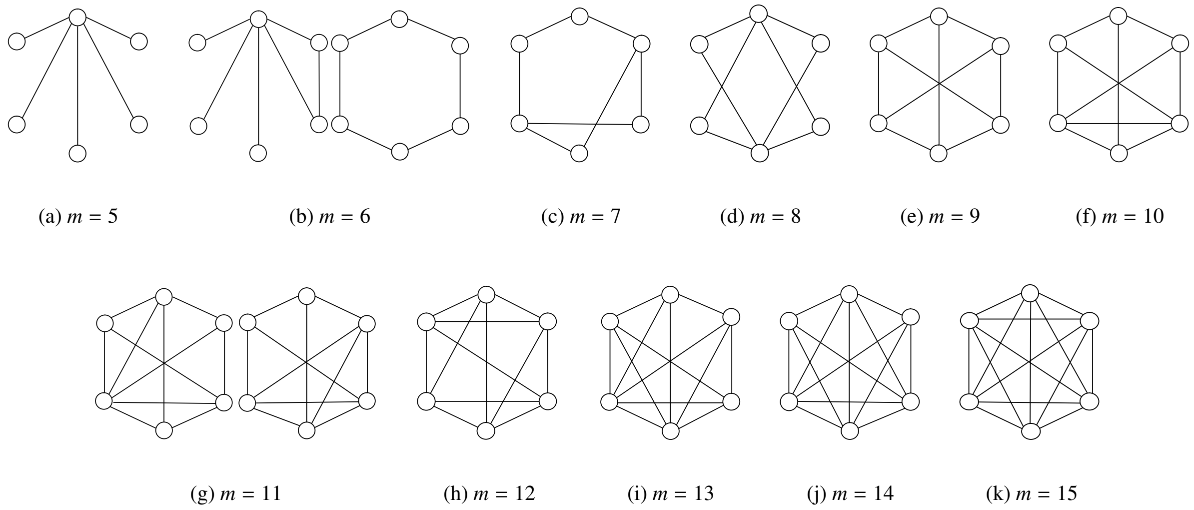


Figure 2: Algebraic connectivity maximizing graphs for $n = 6$. The values of algebraic connectivity are (a) 1, (b) 1, (c) 1.268, (d) 2, (e) 3, (f) 3, (g) 3, (h) 4, (i) 4, (j) 4 and (k) 6.

Lemma 6 ([4]) *Eigenvalues of the Laplacian of the complete bipartite graph $K_{a,n-a}$ are given by*

$$\lambda_i(K_{a,n-a}) = \begin{cases} 0, & i = 1 \\ a, & i = 2, 3, \ldots, a-1 \\ n-a, & i = a, \ldots, n-1 \\ n, & i = n \end{cases}$$

By Lemmas 1 and 6, we have the following theorem.

Theorem 5 *If $a - 2a^2/n < 1$, the complete bipartite graph $K_{a,n-a}$ is an algebraic connectivity maximizing graph in $\mathcal{G}_{n,a(n-a)}$.*

Proof: Let G be any graph in $\mathcal{G}_{n,a(n-a)}$. The average degree of G is given by

$$\frac{2m}{n} = \frac{2a(n-a)}{n} = 2a\left(1 - \frac{a}{n}\right) = a + a - \frac{2a^2}{n}.$$

If $a - 2a^2/n < 1$, this quantity is less than $a + 1$ and hence the minimum degree of G is at most a. By this fact and Lemma 1, we have $\lambda_2(G) \leq \delta(G) \leq a$. On the other hand, we have from Lemma 6 that $\lambda_2(K_{a,n-a}) = a$. Therefore, the complete bipartite graph $K_{a,n-a}$ is an algebraic connectivity maximizing graph in $\mathcal{G}_{n,a(n-a)}$. □

Theorem 6 *The complete bipartite graph $K_{a,n-a}$ is an algebraic connectivity locally maximizing graph in $\mathcal{G}_{n,a(n-a)}$.*

Proof: Let G be any graph in $\mathcal{N}_{n,a(n-a)}(K_{a,n-a})$. It is easily seen that G has at least one node whose degree is a. By this fact and Lemma 1, we have $\lambda_2(G) \leq \delta(G) \leq a$. On the other hand, we have from Lemma 6 that $\lambda_2(K_{a,n-a}) = a$. Therefore, the complete bipartite graph $K_{a,n-a}$ is an algebraic connectivity locally maximizing graph in $\mathcal{G}_{n,a(n-a)}$. □

4. Conclusions

In this paper, we have considered the problem of finding undirected graphs maximizing the algebraic connectivity for the given number of nodes and edges. We first introduced notions of algebraic connectivity maximizing graphs and algebraic connectivity locally maximizing graphs. We then proved some fundamental results about these notions.

Acknowledgments

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References

[1] R. Olfati-Saber, J. A. Fax and R. M. Murray, "Consensus and cooperation in networked multi-agent systems," *Proc. IEEE*, vol.95, no.1, pp.215–233, 2007.

[2] R. Olfati-Saber and R. M. Murray, "Consensus protocols for networks of dynamic agents," in *Proc. 2003 American Control Conference*, pp.951–956, 2003.

[3] M. Fiedler, "Algebraic connectivity of graphs," *Czechoslovak Math. J.*, vol.23, no.98, pp.298–305, 1973.

[4] W. N. Anderson, Jr. and T. D. Morley, "Eigenvalues of the Laplacian of a graph," *Linear and Multilinear Algebra*, vol.18, pp.141–125, 1985.

[5] B. Mohar, "The Laplacian spectrum of graphs" in *Graph Theory, Combinatorics and Applications*, Y. Alavi, G. Chartrand, O. Ollermann and A. Schwenk, Eds. Wiley, New York, pp.871–898, 1991.