2014 International Symposium on Nonlinear Theory and its Applications
NOLTA2014, Luzern, Switzerland, September 14-18, 2014

On Graphs that Locally Maximize Algebraic Connectivity in the Space of Graphs with the Fixed Degree Sequence

Takuro Fujihara† and Norikazu Takahashi†

† Graduate School of Natural Science and Technology, Okayama University
3–1–1 Tsushima-naka, Kita-ku, Okayama 700–8530, Japan
Email: fujihara@momo.cs.okayama-u.ac.jp, takahashi@cs.okayama-u.ac.jp

Abstract—The second smallest eigenvalue of the Laplacian matrix, also known as the algebraic connectivity, is an important measure that characterizes the performance of some dynamic processes on the network. In this paper, we study the problem of finding graphs that locally maximize the algebraic connectivity in the space of graphs with the fixed degree sequence. We first prove that complete bipartite graphs are such graphs. We next find some 3-regular graphs that locally maximize the algebraic connectivity by using a local search algorithm based on 2-switch.

1. Introduction

The algebraic connectivity of a network is defined as the second smallest eigenvalue of the Laplacian matrix of the network [1]. It is well known that the performance of some dynamic processes on the network is characterized by the algebraic connectivity [2]. For example, the convergence rate of the consensus algorithm proposed by Olfati-Saber and Murray [3] for multiagent networks is determined by the algebraic connectivity: the larger the algebraic connectivity is, the faster the consensus algorithm converges. The algebraic connectivity is also an important measure that determines the robustness of networks.

Finding graphs that maximize the algebraic connectivity under certain conditions is a fundamental problem not only from a theoretical but also from a practical point of view. In fact, this problem has recently been studied extensively in various fields from mathematics to engineering [4–7]. Wang *et al.* [4] focused their attention on graphs composed of a chain of some cliques, and proved under some assumptions that these graphs maximize the algebraic connectivity in the set of graphs with the given diameter. Ogiwara *et al.* [5] identified some classes of graphs that maximize or locally maximize the algebraic connectivity when the number of vertices and edges are given. Dai and Mesbahi [6] considered the optimal topology design problem for dynamic networks in three different scenarios, and formulated these problems as mathematical programming problems. Sydney [7] proposed algorithms for rewiring edges in order to maximally increase the algebraic connectivity.

This work was partially supported by KAKENHI 24560076 and 23310104.

In this paper, we consider the problem of finding graphs that locally maximize the algebraic connectivity in the space of graphs with the fixed degree sequence. Note that this problem setting differs from those works mentioned above. We first introduce definitions of the algebraic connectivity maximizing (ACM) graphs and the algebraic connectivity locally maximizing (ACLM) graphs. We then prove that complete bipartite graphs are ACLM graphs. We finally propose a local search algorithm based on 2-switch to find ACLM graphs, and apply it to 3-regular graphs.

2. Algebraic Connectivity Locally Maximizing Graph

Throughout this paper, by a graph, we mean a simple undirected graph. Let $G = (V(G), E(G))$ be a graph composed of n vertices and m edges, where $V(G) = \{1, 2, \ldots, n\}$ is the vertex set and $E(G) = \{e_1, e_2, \ldots, e_m\}$ is the edge set. Each edge is expressed as an unordered pair of two distinct vertices like $\{i, j\}$. The matrix $A(G) = (a_{ij}(G))$ defined by

$$a_{ij}(G) = \begin{cases} 1, & \text{if } \{i, j\} \in E(G), \\ 0, & \text{otherwise}, \end{cases}$$

is called the adjacent matrix of G. The diagonal matrix $D(G) = \mathrm{diag}(d_1(G), d_2(G), \ldots, d_n(G))$ defined by

$$d_i(G) = |\{j \mid \{i, j\} \in E(G)\}|$$

is called the degree matrix of G. The Laplacian matrix $L(G)$ of G is then defined in terms of $A(G)$ and $D(G)$ as

$$L(G) = D(G) - A(G)\,.$$

Let $\lambda_1(G), \lambda_2(G), \ldots, \lambda_n(G)$ be eigenvalues of $L(G)$. Because $L(G)$ is symmetric, these eigenvalues are all real. We thus assume without loss of generality that $\lambda_1(G) \le \lambda_2(G) \le \cdots \le \lambda_n(G)$. Also, because $L(G)$ is diagonally dominant, all eigenvalues are nonnegative. Furthermore, because $L(G)\mathbf{1} = \mathbf{0}$ holds, where $\mathbf{1}$ and $\mathbf{0}$ are the vectors of all ones and all zeros, respectively, 0 is an eigenvalue of $L(G)$. From these observations, we have $\lambda_1(G) = 0$.

The algebraic connectivity [1] is defined as follows.

Definition 1 The algebraic connectivity of a graph G is the second smallest eigenvalue $\lambda_2(G)$ of $L(G)$.

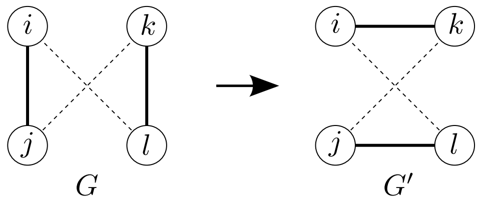


Figure 1: 2-switch. Each dotted line means that there may or may not exist an edge.

Suppose that a graph $G = (V(G), E(G))$ has four distinct vertices such that $\{i, j\} \in E(G)$, $\{k, l\} \in E(G)$, $\{i, k\} \notin E(G)$ and $\{j, l\} \notin E(G)$. Let $G' = (V(G'), E(G'))$ be the graph obtained from G by removing two edges $\{i, j\}$ and $\{k, l\}$, and by adding two new edges $\{i, k\}$ and $\{j, l\}$ (see Fig. 1). This transformation is called 2-switch. It is clear that the degree matrix does not change before and after the application of a 2-switch. Moreover, it is well known that, for any pair of graphs G and G' such that $D(G) = D(G')$, G can be transformed into G' by applying 2-switches sequentially [8].

Let us now give two definitions for those graphs that maximize or locally maximize the algebraic connectivity.

Definition 2 A graph G is called an algebraic connectivity maximizing (ACM) graph in $\mathcal{G}_{D(G)}$ if

$$\forall G' \in \mathcal{G}_{D(G)}, \quad \lambda_2(G) \geq \lambda_2(G'),$$

where $\mathcal{G}_{D(G)}$ is the set of all graphs having the same degree matrix as G.

Definition 3 A graph G is called an algebraic connectivity locally maximizing (ACLM) graph in $\mathcal{G}_{D(G)}$ if

$$\forall G' \in \mathcal{N}_{D(G)}(G), \quad \lambda_2(G) \geq \lambda_2(G'),$$

where $\mathcal{G}_{D(G)}$ is same as Definition 2, and $\mathcal{N}_{D(G)}(G)$ is the set of all graphs obtained from G by applying a single 2-switch.

It is apparent from these definitions that if a graph G is an ACM graph in $\mathcal{G}_{D(G)}$ then G is an ACLM graph in $\mathcal{G}_{D(G)}$. However, the converse is not true.

3. Complete Bipartite Graphs

We first consider whether the complete bipartite graph $K_{a,n-a}$ is an ACLM graph in $\mathcal{G}_{D(K_{a,n-a})}$.

Theorem 1 Let n be any integer greater than or equal to six. Let a be any integer such that

$$2 \leq a \leq \lfloor n/2 \rfloor \tag{1}$$

where $\lfloor n/2 \rfloor$ denotes the largest integer less than or equal to $n/2$. Then the complete bipartite graph $K_{a,n-a}$ is an ACLM graph in $\mathcal{G}_{D(K_{a,n-a})}$.

Proof Let G be any graph in $\mathcal{N}_{D(K_{a,n-a})}(K_{a,n-a})$. We assume without loss of generality that G is obtained from $K_{a,n-a}$ by removing edges $\{1, a+1\}$ and $\{2, a+2\}$ and adding edges $\{1, 2\}$ and $\{a+1, a+2\}$. Let G_1 be the graph obtained from $K_{a,n-a}$ by removing the edge $\{1, a+1\}$. Let G_2 be the graph obtained from G_1 by adding the edge $\{1, 2\}$. Let G_3 be the graph obtained from G_2 by removing the edge $\{2, a+2\}$. Then G is the graph obtained from G_3 by adding the edge $\{a+1, a+2\}$. By the interlace theorem [9], we have the following inequalities:

$$\lambda_2(G_1) \leq \lambda_2(K_{a,n-a}) \leq \lambda_3(G_1) \leq \lambda_3(K_{a,n-a}) \leq \lambda_4(G_1) \leq \lambda_4,$$

$$\lambda_3(G_1) \leq \lambda_3(G_2) \leq \lambda_4(G_1),$$

$$\lambda_3(G_3) \leq \lambda_3(G_2) \leq \lambda_4(G_3),$$

$$\lambda_2(G) \leq \lambda_3(G_3) \leq \lambda_3(G) \leq \lambda_4(G_3) \leq \lambda_4(G).$$

From these inequalities, we have

$$\lambda_2(G) \leq \lambda_3(G_3) \leq \lambda_3(G_2) \leq \lambda_4(G_1) \leq \lambda_4(K_{a,n-a}). \tag{2}$$

Because $a \leq n/2$, the eigenvalues of $L(K_{a,n-a})$ satisfies

$$\lambda_2(K_{a,n-a}) = \cdots = \lambda_{n-a}(K_{a,n-a}) = a$$

as shown in [10]. Moreover, because $n \geq 6$ and (1) hold, we have

$$\lambda_4(K_{a,n-a}) = \lambda_2(K_{a,n-a}). \tag{3}$$

It follows from (2) and (3) that

$$\lambda_2(G) \leq \lambda_4(K_{a,n-a}) = \lambda_2(K_{a,n-a}).$$

This means that $K_{a,n-a}$ is an ACLM graph in $\mathcal{G}_{D(K_{a,n-a})}$ because G is any member of $\mathcal{N}_{D(K_{a,n-a})}(K_{a,n-a})$. □

Although Theorem 1 says that complete bipartite graphs are ACLM graphs if the number of vertices is not less than six, it is not clear whether the algebraic connectivity of $K_{a,n-a}$ with $n \geq 6$ and (1) is strictly greater than that of any graph in $\mathcal{N}_{D(K_{a,n-a})}(K_{a,n-a})$. The following theorem addresses this issue.

Theorem 2 Let n and a be the same as in Theorem 1. For any graph $G \in \mathcal{N}_{D(K_{a,n-a})}(K_{a,n-a})$, we have

$$\lambda_2(G) \leq \lambda_2(K_{a,n-a}) - 1 + \frac{2}{n-a} \leq \lambda_2(K_{a,n-a}) - \frac{1}{3}.$$

Proof We assume without loss of generality that G is the graph obtained from $K_{a,n-a}$ by removing edges $\{1, a+1\}$ and $\{2, a+2\}$ and adding edges $\{1, 2\}$ and $\{a+1, a+2\}$. Then the Laplacian matrix of G is given by

$$L(G) = L(K_{a,n-a}) - M$$

where $M = (m_{ij})$ is given by

$$m_{ij} = \begin{cases} 1, & \text{if } (i,j) = (1,2), (2,1), \\ & \qquad (a+1, a+2), (a+2, a+1), \\ -1, & \text{if } (i,j) = (1, a+1), (a+1, 1), \\ & \qquad (2, a+2), (a+2, 2), \\ 0, & \text{otherwise}. \end{cases}$$

Since $\lambda_2(G)$ is expressed as

$$\lambda_2(G) = \min_{\boldsymbol{x}^T\mathbf{1}=0,\, \|\boldsymbol{x}\|=1} \boldsymbol{x}^T L(G)\boldsymbol{x}\,,$$

we can find an upper bound for $\lambda_2(G)$ as

$$\lambda_2(G) \le \boldsymbol{v}^T L(G)\boldsymbol{v} \tag{4}$$

where $\boldsymbol{v}$ is any vector satisfying $\boldsymbol{v}^T\mathbf{1} = 0$ and $\|\boldsymbol{v}\| = 1$. Let us consider the case where $\boldsymbol{v}$ is set to an eigenvector of $K_{a,n-a}$ associated with $\lambda_2(K_{a,n-a}) = a$. By solving the equation $L(K_{a,n-a})\boldsymbol{v} = a\boldsymbol{v}$, we have

$$\sum_{i=1}^{a} v_i = 0 \;\text{ and }\; \sum_{i=a+1}^{n} v_i = 0 \tag{5}$$

if $a = n/2$, and

$$v_1 = v_2 = \cdots = v_a = 0 \;\text{ and }\; \sum_{i=a+1}^{n} v_i = 0 \tag{6}$$

if $a < n/2$. Because (6) implies (5), we assume hereafter that $\boldsymbol{v}$ satisfies (6) as well as $\|\boldsymbol{v}\| = 1$. Then the right-hand side of (4) can be expressed as

$$\begin{aligned} \boldsymbol{v}^T L(G)\boldsymbol{v} &= \boldsymbol{v}^T(L(K_{a,n-a}) - M)\boldsymbol{v} \\ &= \lambda_2(K_{a,n-a}) - \boldsymbol{v}^T M\boldsymbol{v} \\ &= \lambda_2(K_{a,n-a}) - 2v_{a+1}v_{a+2}\,. \end{aligned}$$

We now consider the problem of minimizing the second term subject to (6) and $\|\boldsymbol{v}\| = 1$. This problem is formulated as the optimization problem:

$$\begin{array}{ll} \text{minimize} & -2v_{a+1}v_{a+2} \\ \text{subject to} & \sum_{i=a+1}^{n} v_i = 0\,, \\ & \sum_{i=a+1}^{n} v_i^2 = 1\,. \end{array} \tag{7}$$

Using the method of Lagrange multiplier, we obtain an optimal solution of (7) which is given by

$$v_i^* = \begin{cases} \frac{\mu_1}{2(1-\mu_2)}, & \text{if } i = a+1, a+2\,, \\ -\frac{\mu_1}{2\mu_2}, & \text{if } i = a+3, a+4, \ldots, n \end{cases} \tag{8}$$

where

$$\mu_1 = \sqrt{\frac{8(n-a-2)}{(n-a)^3}}, \quad \mu_2 = \frac{n-a-2}{n-a}\,.$$

Substituting (8) to the objective function of (7) and using the assumptions on n and a, we have

$$\begin{aligned} -2v_{a+1}^* v_{a+2}^* &= -1 + 2/(n-a) \\ &\le -1 + 4/n \quad (\because a \le n/2) \\ &\le -1 + 2/3 \quad (\because n \ge 6) \\ &= -1/3 \end{aligned}$$

which completes the proof. □

4. Local Search Algorithm for Finding ACLM Graphs

In this section, we consider the problem of finding an ACLM graph in $\mathcal{G}_{D(G_0)}$ for a given G_0. Because it is very difficult to solve this problem analytically, we use a simple 2-switch-based local search algorithm described below.

Algorithm 1 Given a graph G_0, the following algorithm returns an ACLM graph $G \in \mathcal{G}_{D(G_0)}$.

1. Set $t \leftarrow 0$.
2. Set $(i, j, k, l) \leftarrow (1, 2, 3, 4)$.
3. If 2-switch is applicable to four vertices i, j, k and l, that is, if $\{i, j\} \in E(G_t)$, $\{k, l\} \in E(G_t)$, $\{i, k\} \notin E(G_t)$ and $\{j, l\} \notin E(G_t)$ are satisfied then go to Step 4. Otherwise, go to Step 5.
4. Apply 2-switch to four vertices i, j, k and l of G_t to get G_t'. If $\lambda_2(G_t') > \lambda_2(G_t)$ then set $G_{t+1} \leftarrow G_t'$ and $t \leftarrow t + 1$ and go to Step 2. Otherwise, go to Step 5.
5. If $(i, j, k, l) = (n-3, n-2, n-1, n)$ then return G_t and stop. Otherwise, update (i, j, k, l) in ascending lexicographic order and go to Step 3.

Note that the variable (i, j, k, l) is reset to the initial value $(1, 2, 3, 4)$ as soon as a graph G_t' such that $\lambda_2(G_t') > \lambda_2(G_t)$ is found. Therefore, this algorithm stops only when no such G_t' which means that the output is an ACLM graph.

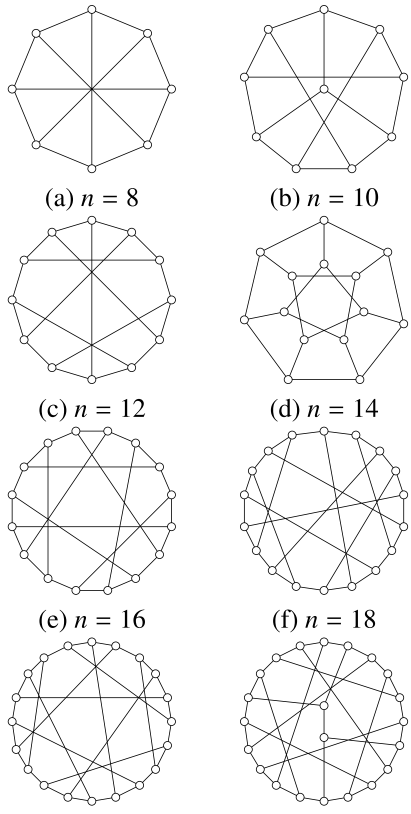
Although Algorithm 1 can be applied to any graph, we hereafter focus our attention on 3-regular graphs. In multi-agent networks, it is very natural to assume that all agents have the same communication capability. Because the degree of each vertex representing an agent can be considered as the communication capability of the agent, 3-regular graphs correspond to the simplest case in which every agent can interact with three other agents.

For each $n \in \{8, 10, \ldots, 22\}$, we choose a 3-regular graph with n vertices as the initial graph G_0 and apply Algorithm 1 to obtain an ACLM graph in $\mathcal{G}_{D(G_0)}$. Apparently, how to choose the initial graph G_0 is an important issue. In our experiment, we set G_0 to the circulant graph such that the first row of the adjacency matrix $A(G_0) = (a_{ij}(G_0))$ is given by

$$a_{1j}(G_0) = \begin{cases} 1, & \text{if } j = 2, \frac{n}{2} + 1, n\,, \\ 0, & \text{otherwise}\,. \end{cases}$$

For $n = 6$, the circulant graph having this kind of adjacent matrix is the bipartite graph $K_{3,3}$ which is an ACLM graph in $\mathcal{G}_{D(K_{3,3})}$ as proved by Theorem 1. So the algebraic connectivity of the circulant graph is expected to be high.

ACLM graphs obtained by Algorithm 1 are shown in Fig. 2. First of all, we should note that in all cases except (a) the obtained graph differs from the initial one. This means that the circulant graph used for the initial graph is not an ACLM graph in general. Second, we have to say that it is difficult to find common features for all graphs. Further investigation is needed to better understand ACLM graphs in 3-regular graphs.



(a) $n = 8$ (b) $n = 10$ (c) $n = 12$ (d) $n = 14$ (e) $n = 16$ (f) $n = 18$ (g) $n = 20$ (h) $n = 22$

Figure 2: ACLM graphs obtained by Algorithm 1.

The values of the algebraic connectivities of the initial graph, denoted by G_0, and the final graph, denoted by G_∞, are shown in Table 1. For all cases except (a), the algebraic connectivity was increased through the local search by a factor greater than 1.4. In addition, the factor is monotone increasing with the number of vertices.

5. Conclusion

The problem of finding graphs that maximize the algebraic connectivity in the space of graphs with the fixed degree sequence has been studied in this paper. First, we have proved that complete bipartite graphs have this property. Next, we have presented a local search algorithm based on 2-switch to find graphs with the same property and applied to 3-regular graphs. Further investigation on 3-regular graphs is a future problem.

Table 1: Algebraic connectivities of the initial and final graphs. For each n, G_0 and G_∞ represent the initial and final graph, respectively.

n	$\lambda_2(G_0)$	$\lambda_2(G_\infty)$	$\lambda_2(G_\infty)/\lambda_2(G_0)$
8	2	2	1
10	1.381966	2	1.4472136
12	1	1.4679111	1.4679111
14	0.7530204	1.2891708	1.7119998
16	0.5857864	1.1729091	2.0022812
18	0.4679111	1.0303845	2.2020946
20	0.2266939	0.8299135	3.6609432
22	0.1920181	0.7406081	3.8569703

References

[1] M. Fiedler, "Algebraic connectivity of graphs," *Czechoslovak Mathematical Journal*, vol. 23, no. 98, pp. 298–305, 1973.

[2] M. Mesbahi and M. Egerstedt, *Graph Theoretic Methods in Multiagent Networks*. Princeton: Princeton University Press, 2010.

[3] R. Olfati-Saber and R. M. Murray, "Consensus protocols for networks of dynamic agents," in *Proceedings of the 2003 American Control Conference*, pp. 951–956, June 2003.

[4] H. Wang, R. Kooij, and P. Van Mieghem, "Graphs with given diameter maximizing the algebraic connectivity," *Linear Algebra and its Applications*, vol. 433, pp. 1889–1908, 2010.

[5] K. Ogiwara and N. Takahashi, "On topology of networked multi-agent systems for fast consensus," in *Proceedings of 2011 International Symposium on Nonlinear Theory and its Applications*, pp. 56–59, September 2011.

[6] R. Dai and M. Mesbahi, "Optimal topology design for dynamic networks," in *Proceedings of the 50th IEEE Conference on Decision and Control and European Control Conference*, pp. 1280–1285, December 2011.

[7] A. Sydney, C. Scoglio, and D. Gruenbacher, "Optimizing algebraic connectivity by edge rewiring," *Applied Mathematics and Computation*, vol. 219, pp. 5465–5479, 2013.

[8] S. Bereg and H. Ito, "Transforming graphs with the same degree sequence," in *Computational Geometry and Graph Theory*, pp. 25–32, Springer, 2008.

[9] C. D. Godsil, G. Royle, and C. Godsil, *Algebraic graph theory*. Springer New York, 2001.

[10] W. N. Anderson, Jr and T. D. Morley, "Eigenvalues of the Laplacian of a graph," *Linear and Multilinear Algebra*, vol. 18, pp. 141–145, 1985.