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Extensions of a Theorem on Algebraic Connectivity Maximizing Graphs

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Abstract—The second smallest eigenvalue of the Laplacian matrix of a graph, also known as the algebraic connectivity, is an important measure that represents how strongly the graph is connected. The algebraic connectivity also characterizes the performance of some dynamic processes on networks such as consensus in multiagent networks and synchronization of coupled oscillators. In this paper, we study the problem of finding graphs that maximize the algebraic connectivity among all graphs with the same number of vertices and edges, and extend a known result about complete bipartite graphs to complete multipartite graphs.

1. Introduction

Algebraic connectivity [1] of a graph, which is defined as the second smallest eigenvalue of the Laplacian matrix, is an important measure that represents how strongly the graph is connected. Not only it has been intensively studied in mathematics [2, 3, 4], but also it has attracted a great deal of attention from researchers in engineering. For example, the convergence rate of a well-known consensus algorithm for multiagent networks is determined by the algebraic connectivity of the network [5]. Also, the algebraic connectivity plays important roles in the design of computer networks [6] and air transportation networks [7].

Recently, Ogiwara *et al.* [8] studied the problem of finding graphs with a given number of vertices and edges that maximize the algebraic connectivity. This problem is important from various perspectives such as the fast convergence of the consensus algorithm, the robustness of networks against failures and attacks, and so on. They proved that some well-known classes of graphs such as star graphs, cycle graphs and complete bipartite graphs are algebraic connectivity maximizers under certain conditions. This problem was also considered by Kolokolnikov [9]. He presented a conjecture that the complete bipartite graph $K_{2,n-2}$ maximizes the algebraic connectivity among all graphs with n vertices and $2(n-2)$ edges. He also showed by exhaustive search that this conjecture holds true for all n less than or equal to 13. Fujihara and Takahashi [10] studied a slightly different problem and proved that any complete multipartite graph maximizes the algebraic connectivity among all graphs with the same degree matrix.

In this paper, we prove that if a complete multipartite graph satisfies a certain condition then it maximizes the algebraic connectivity among all graphs with the same number of vertices and edges. This is an extension of a theorem given by Ogiwara *et al.* [8], which states that any complete bipartite graph K_{n_1,n_2} with $n_1 \approx n_2$ maximizes the algebraic connectivity among all graphs with n_1+n_2 vertices and $n_1 n_2$ edges. We further generalize this result to graphs obtained from complete multipartite graphs by adding some edges.

2. Algebraic Connectivity Maximizing Graphs

2.1. Notations and Definitions

Throughout this paper, by a graph we mean a simple undirected graph. Let $G = (V(G), E(G))$ be a graph with the vertex set $V(G) = \{1, 2, \ldots, n\}$ and the edge set $E(G)$. The Laplacian matrix of G is defined by $\boldsymbol{L}(G) = \boldsymbol{D}(G) - \boldsymbol{A}(G)$ [1] where $\boldsymbol{D}(G) = \mathrm{diag}(d_1(G), d_2(G), \ldots, d_n(G))$ is the degree matrix and $\boldsymbol{A}(G) = (a_{ij}(G)) \in \{0, 1\}^{n\times n}$ is the adjacency matrix. Because $\boldsymbol{L}(G)$ is positive semi-definite, its eigenvalues, which are denoted by $\lambda_1(G), \lambda_2(G), \ldots, \lambda_n(G)$, are nonnegative real numbers. In the remainder of this paper, we assume without loss of generality that $0 \leq \lambda_1(G) \leq \lambda_2(G) \leq \cdots \leq \lambda_n(G)$. Because the Laplacian matrix satisfies $\boldsymbol{L}(G)\boldsymbol{1} = \boldsymbol{0} = 0 \cdot \boldsymbol{1}$ where $\boldsymbol{1}$ is the vector of all ones and $\boldsymbol{0}$ is the zero vector, the smallest eigenvalue $\lambda_1(G)$ is 0 and $\boldsymbol{1}$ is an eigenvector associated with $\lambda_1(G)$. The second smallest eigenvalue $\lambda_2(G)$ is called the algebraic connectivity [1] of G. It represents how strongly the graph is connected. In particular, $\lambda_2(G)$ is positive if and only if G is connected.

The algebraic connectivity maximizing graph is defined as follows [8].

Definition 1 Let $\mathcal{G}_{n,m}$ be the set of all graphs with n vertices and m edges. If a graph $G \in \mathcal{G}_{n,m}$ satisfies the condition that

$$\forall G' \in \mathcal{G}_{n,m}, \quad \lambda_2(G) \geq \lambda_2(G')$$

then G is called an algebraic connectivity maximizing graph in $\mathcal{G}_{n,m}$.

2.2. Known Results

If the vertex set $V(G) = \{1, 2, \ldots, n\}$ of a graph G is partitioned into $k\,(\geq 2)$ subsets $V_1, V_2, \ldots, V_k$ in such a way that vertices $i \in V_a$ and $j \in V_b$ are adjacent to each other if and only if $a \neq b$, then G is called a complete k-partite graph and denoted by $K_{n_1,n_2,\ldots,n_k}$ where $n_l = |V_l|$ for $l = 1, 2, \ldots, k$. An example of such a graph is shown in Fig. 1.

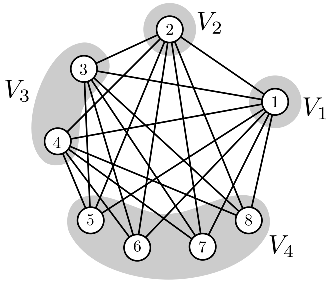

Figure 1: Complete 4-partite graph $K_{1,1,2,4}$.

In the following discussions, we assume without loss of generality that

$$1 \le n_1 \le n_2 \le \cdots \le n_k \,. \tag{1}$$

The complete n-partite graph $K_{1,1,\ldots,1}$ is called the complete graph and denoted by K_n in this paper.

First, we present some fundamental results about the eigenvalues of the Laplacian matrix and their multiplicities.

Lemma 1 ([2]) The eigenvalues of $\boldsymbol{L}(K_n)$ are 0, with multiplicity 1, and n, with multiplicity $n-1$.

Theorem 1 ([2]) If λ is an eigenvalue of $\boldsymbol{L}(G)$ then $0 \le \lambda \le n$. The multiplicity of 0 equals the number of connected components of G. The multiplicity of n equals one less than the number of connected components of G^{c}, the complement of G.

Theorem 2 The eigenvalues of $\boldsymbol{L}(K_{n_1,n_2,\ldots,n_k})$ are $0, n-n_k, n-n_{k-1}, \ldots, n-n_1$ and n, with multiplicity $1, n_k-1, n_{k-1}-1, \ldots, n_1-1$ and $k-1$, respectively.

Proof: Because $K_{n_1,n_2,\ldots,n_k}$ is connected, it follows from Theorem 1 that the smallest eigenvalue 0 of $\boldsymbol{L}(K_{n_1,n_2,\ldots,n_k})$ has multiplicity 1. Also, because the complement $K^{\mathrm{c}}_{n_1,n_2,\ldots,n_k}$ of $K_{n_1,n_2,\ldots,n_k}$ has k connected components which are isomorphic to $K_{n_1}, K_{n_2}, \ldots, K_{n_k}$, it follows from Theorem 1 that $\boldsymbol{L}(K_{n_1,n_2,\ldots,n_k})$ has the largest eigenvalue n with multiplicity $k-1$. Moreover, we see from Lemma 1 that the eigenvalues of $\boldsymbol{L}(K^{\mathrm{c}}_{n_1,n_2,\ldots,n_k})$ are $0, n_1, n_2, \ldots, n_k$ with multiplicity $k, n_1-1, n_2-1, \ldots, n_k-1$, respectively. Hence, in order to complete the proof, we only have to show that if $\boldsymbol{L}(K^{\mathrm{c}}_{n_1,n_2,\ldots,n_k})$ has an eigenvalue λ other than 0 and n with multiplicity μ then $\boldsymbol{L}(K_{n_1,n_2,\ldots,n_k})$ has an eigenvalue $n-\lambda$ with the same multiplicity.

Note that $K_{n_1,n_2,\ldots,n_k}$ and $K^{\mathrm{c}}_{n_1,n_2,\ldots,n_k}$ satisfy

$$\boldsymbol{L}(K_{n_1,n_2,\ldots,n_k}) = \boldsymbol{L}(K_n) - \boldsymbol{L}(K^{\mathrm{c}}_{n_1,n_2,\ldots,n_k}) \,. \tag{2}$$

Let $\boldsymbol{v}$ be an eigenvector of $\boldsymbol{L}(K^{\mathrm{c}}_{n_1,n_2,\ldots,n_k})$ associated with λ. Then $\boldsymbol{v}$ is orthogonal to $\boldsymbol{1}$. Multiplying both sides of (2) by $\boldsymbol{v}$ from right, we have

$$\begin{aligned}(\boldsymbol{L}(K_n) - \boldsymbol{L}(K^{\mathrm{c}}_{n_1,n_2,\ldots,n_k}))\boldsymbol{v} &= \left(n\boldsymbol{I} - \boldsymbol{1}\boldsymbol{1}^{\mathrm{T}}\right)\boldsymbol{v} - \lambda\boldsymbol{v} \\ &= n\boldsymbol{v} - \lambda\boldsymbol{v} \\ &= (n-\lambda)\boldsymbol{v} \end{aligned} \tag{3}$$

where $\boldsymbol{I}$ is the identity matrix. From (2) and (3) we have

$$\boldsymbol{L}(K_{n_1,n_2,\ldots,n_k})\boldsymbol{v} = (n-\lambda)\boldsymbol{v}$$

which means that $n-\lambda$ is an eigenvalue of $\boldsymbol{L}(K_{n_1,n_2,\ldots,n_k})$ and $\boldsymbol{v}$ is an eigenvector associated with $n-\lambda$. In addition, it is easy to see that the dimension of the eigenspace of $\boldsymbol{L}(K^{\mathrm{c}}_{n_1,n_2,\ldots,n_k})$ associated with λ is equal to that of $\boldsymbol{L}(K_{n_1,n_2,\ldots,n_k})$ associated with $n-\lambda$. □

Theorem 2 may not be new. However, it is difficult to find this result in the existing literature. We therefore have provided a proof of it.

Next, we present two results given by Ogiwara *et al.* [8] about the sufficient condition for a complete bipartite graph to be an algebraic connectivity maximizing graph.

Theorem 3 ([8]) If $k=2$ and two positive integers n_1 and n_2 satisfy $n_1+n_2 \ge 3$ and

$$n_1 - \frac{2n_1^2}{n_1+n_2} < 1$$

as well as (1) then the complete bipartite graph K_{n_1,n_2} is an algebraic connectivity maximizing graph in $\mathcal{G}_{n_1+n_2,n_1n_2}$.

Corollary 1 ([8]) If $k=2$ and two positive integers n_1 and n_2 satisfy $n_1+n_2 \ge 3$ and

$$\left\lfloor \frac{n_1+n_2-1}{2} \right\rfloor \le n_1 \le \left\lfloor \frac{n_1+n_2}{2} \right\rfloor$$

as well as (1) then the complete bipartite graph K_{n_1,n_2} is an algebraic connectivity maximizing graph in $\mathcal{G}_{n_1+n_2,n_1n_2}$.

We also provide two well-known results about the algebraic connectivity, that will be needed in later discussions.

Theorem 4 ([1]) If G is not a complete graph then $\lambda_2(G) \le \delta(G)$ where $\delta(G) = \min_{1\le i\le n}\{d_i(G)\}$.

Theorem 5 ([3]) Let $G' \in \mathcal{G}_{n,m+1}$ be a graph obtained by adding an edge to $G \in \mathcal{G}_{n,m}$. Then we have

$$\lambda_1(G) \le \lambda_1(G') \le \lambda_2(G) \le \lambda_2(G') \le \cdots \le \lambda_n(G) \le \lambda_n(G') \,.$$

3. Exhaustive Search of Algebraic Connectivity Maximizing Graphs

In order to see what kind of complete multipartite graphs can be algebraic connectivity maximizing graphs, we developed an exhaustive search algorithm based on the graph enumeration algorithm proposed by Sato and Nakano [11] and applied it to $\mathcal{G}_{n,m}$ for various values of (n,m).

We first applied the algorithm to $\mathcal{G}_{n,m}$ with $n \le 10$ such that it contains a complete bipartite graph. For $n=10$, for example, the algorithm was applied to $\mathcal{G}_{10,9}$, $\mathcal{G}_{10,16}$, $\mathcal{G}_{10,21}$, $\mathcal{G}_{10,24}$ and $\mathcal{G}_{10,25}$. As a result, it was found that any bipartite graph K_{n_1,n_2} with $n_1+n_2 \le 10$ is an algebraic connectivity maximizing graph in $\mathcal{G}_{n_1+n_2,n_1n_2}$.

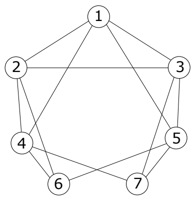

Figure 2: An algebraic connectivity maximizing graph in $\mathcal{G}_{7,14}$.

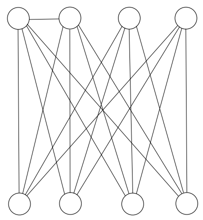

Figure 3: An algebraic connectivity maximizing graph found by the exhaustive search algorithm for $\mathcal{G}_{8,17}$

Next we applied the algorithm to $\mathcal{G}_{n,m}$ such that a complete tripartite graph is contained in it. First let us consider $\mathcal{G}_{7,14}$ which contains $K_{1,2,4}$. An algebraic connectivity maximizing graph found by the algorithm is shown in Fig. 2. Note that it is not a complete multipartite graph. Moreover, all other graphs found by the algorithm were isomorphic to the graph in Fig. 2. This means that $K_{1,2,4}$ is not an algebraic connectivity maximizing graph. On the other hand, for $\mathcal{G}_{8,24}$ and $\mathcal{G}_{9,27}$, the algorithm found $K_{2,2,2,2}$ and $K_{3,3,3,3}$, respectively, as algebraic connectivity maximizing graphs. From these results and Corollary 1, it is conjectured that the complete k-partite graph $K_{n_1,n_2,\ldots,n_k}$ with $n_1 = n_2 = \cdots = n_k$ is an algebraic connectivity maximizing graph. It is proved in the next section that the conjecture is in fact true.

We also applied the algorithm to $\mathcal{G}_{n,m}$ which does not necessarily contain a complete multipartite graph. An algebraic connectivity maximizing graph found for $\mathcal{G}_{8,17}$ is shown in Fig. 3 and that for $\mathcal{G}_{9,28}$ is shown in Fig. 4. It is easily seen that each of them is obtained from a complete multipartite graph $K_{n_1,n_2,\ldots,n_k}$ with $n_1 = n_2 = \cdots = n_k$ by adding one edge. It is proved in the next section that these graphs are algebraic connectivity maximizing graphs.

4. Theoretical Analysis

We give two theorems that can be considered as extensions of Theorem 3 and Corollary 1. Before doing so, we present two lemmas.

Lemma 2 If $k \geq 2$ and k positive integers $n_1, n_2, \ldots, n_k$ satisfy

$$n_k \geq 2 \quad \text{and} \quad n_k \sum_{l=1}^{k-1} n_l \leq \sum_{l=1}^{k-1} n_l^2 \tag{4}$$

as well as (1) then the complete k-partite graph $K_{n_1,n_2,\ldots,n_k}$ is

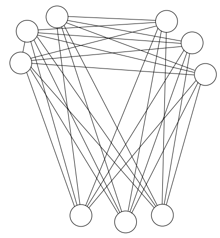

Figure 4: An algebraic connectivity maximizing graph found by the exhaustive search algorithm for $\mathcal{G}_{9,28}$.

an algebraic connectivity maximizing graph in $\mathcal{G}_{n,m}$ where $n = \sum_{l=1}^{k} n_l$ and $m = \sum_{l=1}^{k} n_l(n - n_l)/2$.

Proof: By Theorem 2, the algebraic connectivity of the complete k-partite graph $K_{n_1,n_2,\ldots,n_k}$ is equal to $n - n_k$. We therefore prove under the assumption (4) that $\lambda_2(G) \leq n - n_k$ for all $G \in \mathcal{G}_{n,m}$ where $n = \sum_{l=1}^{k} n_l$ and $m = \sum_{l=1}^{k} n_l(n - n_l)/2$. Furthermore, by Theorem 4, it suffices for us to show under the assumption (4) that $\delta(G) \leq n - n_k$ for all $G \in \mathcal{G}_{n,m}$ (note that G is not a complete graph because of the assumption $n_k \geq 2$). The sum of the degrees of all vertices of G is given by

$$\sum_{i=1}^{n} d_i(G) = 2m = \sum_{l=1}^{k} n_l(n - n_l) = n^2 - \sum_{l=1}^{k} n_l^2.$$

Here it follows from assumption (4) that

$$-\sum_{l=1}^{k} n_l^2 = -\sum_{l=1}^{k-1} n_l^2 - n_k^2 \leq -n_k \sum_{l=1}^{k-1} n_l - n_k^2 = -nn_k$$

from which we have

$$\sum_{i=1}^{n} d_i(G) \leq n^2 - nn_k = n(n - n_k).$$

Therefore, we finally have

$$\delta(G) \leq \frac{1}{n} \sum_{i=1}^{n} d_i(G) = n - n_k$$

which completes the proof. □

Lemma 3 Let k be any integer greater than or equal to 2. Positive integers $n_1, n_2, \ldots, n_k$ satisfy (1) and (4) if and only if $n_1 = n_2 = \cdots = n_k \geq 2$.

Proof: It follows from (1) that $n_k n_l \geq n_l^2$ for $l = 1, 2, \ldots, k-1$. Hence (4) holds if and only if $2n_l \leq n_k n_l = n_l^2$ for $l = 1, 2, \ldots, k-1$, that is, $n_1 = n_2 = \cdots = n_k \geq 2$. □

From Lemmas 2 and 3, we immediately obtain the following theorem.

Theorem 6 If $k \geq 2$, the positive integers $n_1, n_2, \ldots, n_k$ are equal to each other, and $n_1 \geq 2$ then the complete k-partite graph $K_{n_1,n_2,\ldots,n_k}$ is an algebraic connectivity maximizing graph in $\mathcal{G}_{n,m}$ where $n = kn_1$ and $m = kn_1(n - n_1)/2$.

This theorem can be further extended as follows.

Theorem 7 If $k \geq 2$, the positive integers $n_1, n_2, \ldots, n_k$ are equal to each other, $n_1 \geq 2$, and p is a positive integer less than $kn_1/2$ then any graph obtained from the complete k-partite graph $K_{n_1,n_2,\ldots,n_k}$ by adding p edges has the same algebraic connectivity as $K_{n_1,n_2,\ldots,n_k}$ and is an algebraic connectivity maximizing graph in $\mathcal{G}_{n,m+p}$ where $n = kn_1$ and $m = n(n-n_1)/2$.

Proof: Suppose that k, $n_1, n_2, \ldots, n_k$ and p satisfy the assumptions of the statement. We first show that

$$p \leq k(n_1 - 1) - 1. \tag{5}$$

If kn_1 is even then we have

$$\begin{aligned} k(n_1-1)-1-p &\geq k(n_1-1)-1-\left(\frac{kn_1}{2}-1\right) \\ &= \frac{k}{2}(n_1-2) \end{aligned}$$

which is nonnegative. If kn_1 is odd then we have

$$\begin{aligned} k(n_1-1)-1-p &\geq k(n_1-1)-1-\frac{kn_1-1}{2} \\ &= \frac{k(n_1-2)-1}{2} \end{aligned}$$

which is positive because $k \geq 3$ and $n_1 \geq 3$. Therefore, p always satisfies (5). Let G^* be any graph obtained from the complete k-partite graph $K_{n_1,n_1,\ldots,n_1}$ by adding p edges. The eigenvalues of $\boldsymbol{L}(K_{n_1,n_1,\ldots,n_1})$ are 0 with multiplicity 1, $n-n_1$ with multiplicity $k(n_1-1)$, and n with multiplicity $k-1$ due to Theorem 2. By this fact, together with Theorem 5 and (5), we have $\lambda_2(G^*) = n - n_1 = \lambda_2(K_{n_1,n_2,\ldots,n_k})$. In order to prove the second part, it suffices to show that $\lambda_2(G) \leq n-n_1$ for any $G \in \mathcal{G}_{n,m+p}$. By Theorem 4, we have

$$\lambda_2(G) \leq \delta(G) \leq \frac{1}{n}\sum_{i=1}^{n} d_i(G) = \frac{2(m+p)}{n} \tag{6}$$

where

$$\frac{2(m+p)}{n} = \frac{n(n-n_1)+2p}{n} = n - n_1 + \frac{2p}{n}$$

and $2p/n$ is less than 1 from the assumption. Therefore, the inequality (6) implies that $\lambda_2(G) \leq n - n_1$. □

5. Conclusion

In this paper, we first proved that any complete multipartite graph $K_{n_1,n_2,\ldots,n_k}$ with $n_1 = n_2 = \cdots = n_k$ is an algebraic connectivity maximizing graph. We then extended this result to graphs obtained from such complete multipartite graphs by adding some edges. However, we have to say that these results are rather severe, because Theorems 6 or 7 apply to the set of graphs with n vertices and m edges for relatively few choices of n and m. A future problem is to obtain milder sufficient conditions.

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